Signals and Systems

Fall 2003

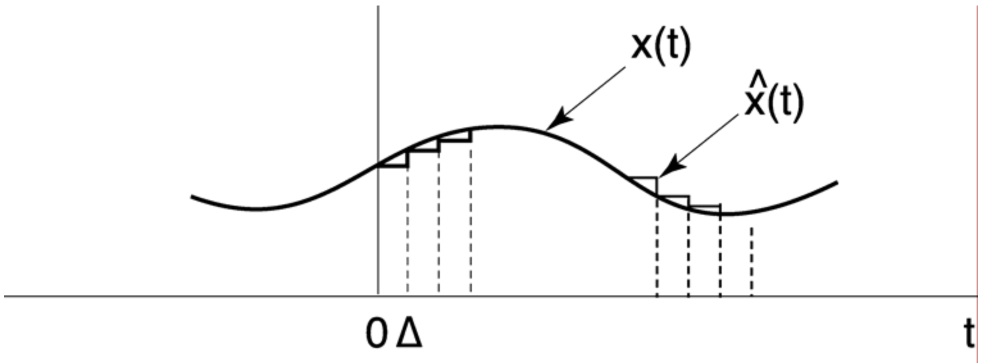
Lecture #4

16 September 2003

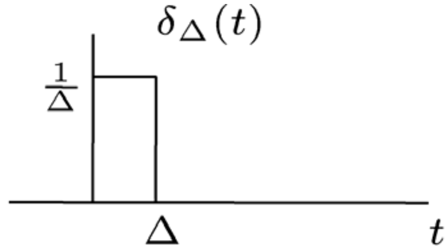
1. Representation of CT Signals in terms of shifted unit impulses
2. Convolution integral representation of CT LTI systems
3. Properties and Examples
4. The unit impulse as an idealized pulse that is "short enough": The operational definition of $\delta(t)$

Representation of CT Signals

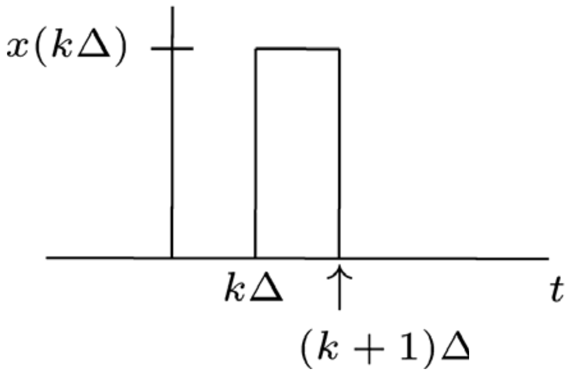
- Approximate any input $x(t)$ as a sum of shifted, scaled pulses



$$\hat{x}(t) = x(k\Delta)\ ,\ k\Delta < t < (k+1)\Delta$$



$\delta_\Delta(t)$ has unit area



$$= \quad x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\Downarrow$$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\downarrow \text{ limit as } \Delta \to 0$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The Sifting Property of the Unit Impulse

Response of a CT LTI System

$$\delta_\Delta(t) \longrightarrow h_\Delta(t)$$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta$$

$$\Downarrow$$

Impulse response: $\boxed{\delta(t) \longrightarrow h(t)}$

Taking limits $\Delta \rightarrow 0$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \underbrace{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}_{\text{Convolution Integral}}$$

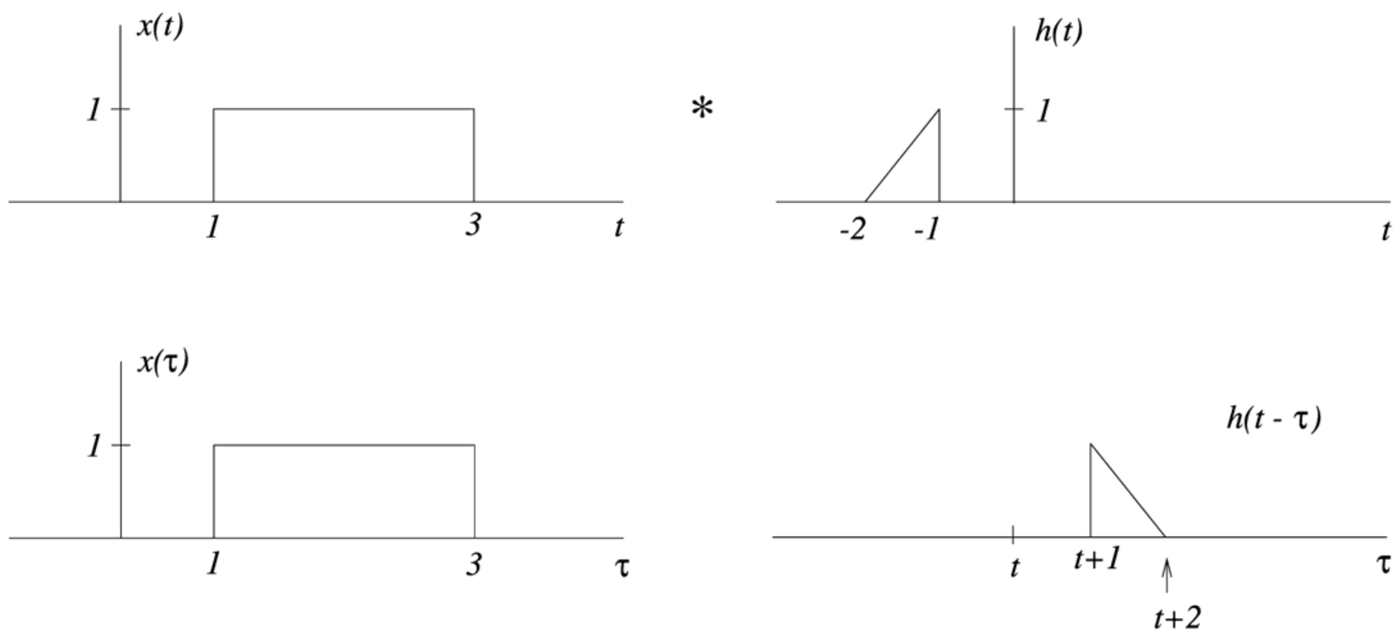
Operation of CT Convolution

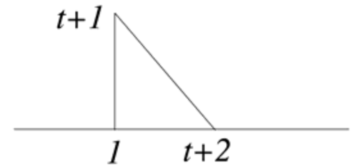
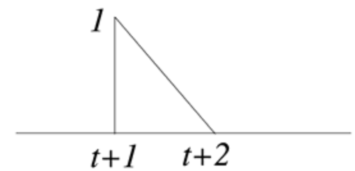
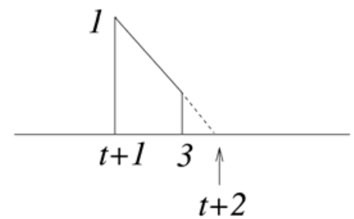
$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

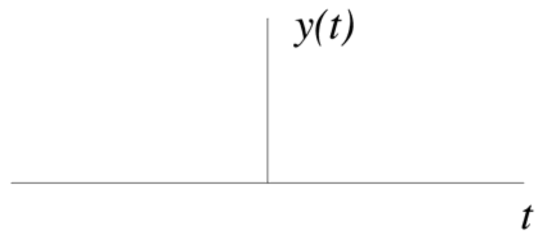
$$h(\tau) \xrightarrow{\text{Flip}} h(-\tau) \xrightarrow{\text{Slide}} h(t-\tau)$$

$$\xrightarrow{\text{Multiply}} x(\tau)h(t-\tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Example: **CT convolution**



Time Interval	$x(\tau) \cdot h(t-\tau)$		Output
$t < -1$	0	$\Longrightarrow$	$y(t) = 0$
$-1 < t < 0$		$\Longrightarrow$	$y(t) = \frac{1}{2}(t+2)(t+2-1) = \frac{1}{2}(t+1)^2$
$0 < t < 1$		$\Longrightarrow$	$y(t) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$
$1 < t < 2$		$\Longrightarrow$	$y(t) = \frac{1}{2} - \frac{1}{2}(t+2-3)(t-1) = \frac{1}{2} - \frac{1}{2}(t-1)^2$
$t > 2$	0	$\Longrightarrow$	$y(t) = 0$



PROPERTIES AND EXAMPLES

1) Commutativity: $$x(t) * h(t) = h(t) * x(t)$$

2) $$x(t) * \delta(t - t_0) = x(t - t_0)$$ Sifting property: $x(t) * \delta(t) = x(t)$

3) An integrator: $$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

So if input $x(t) = \delta(t)$
output $y(t) = h(t)$

$$\Downarrow$$

$$h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

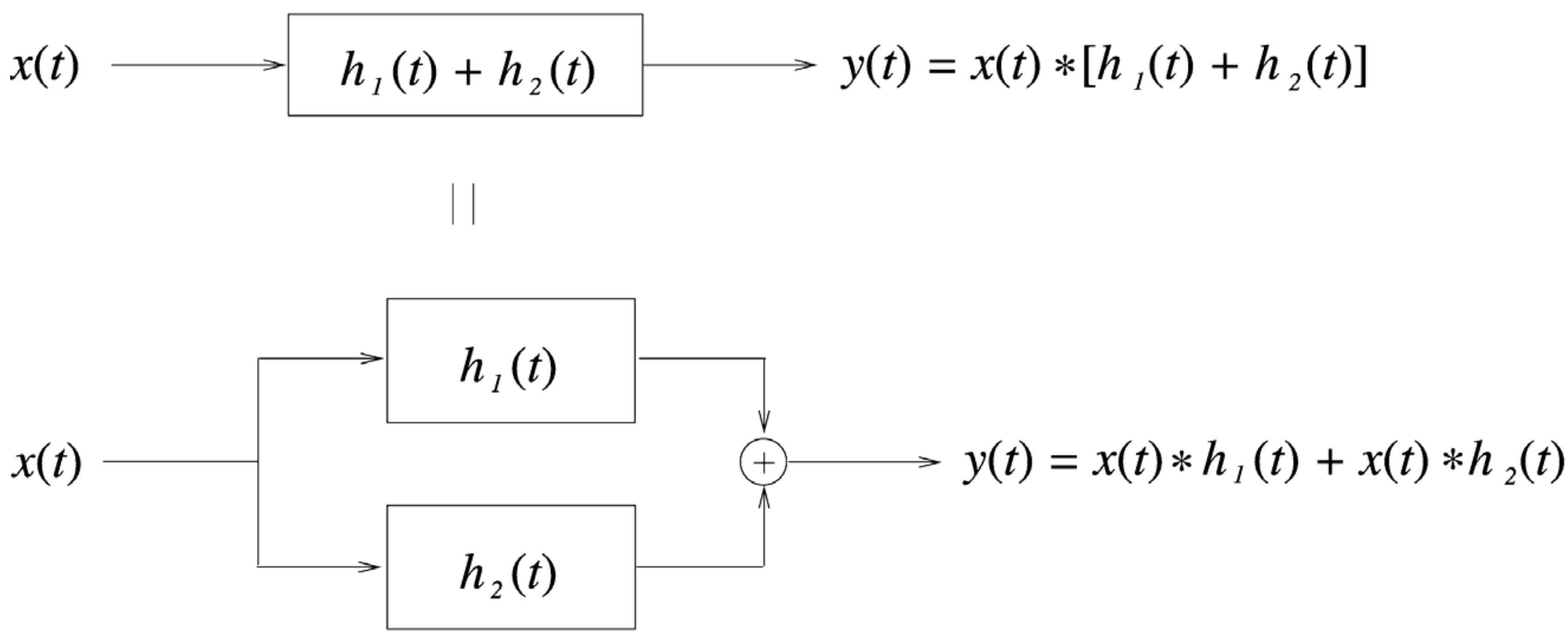
That is

$$y(t) = x(t) * h(t) = \boxed{x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau}$$

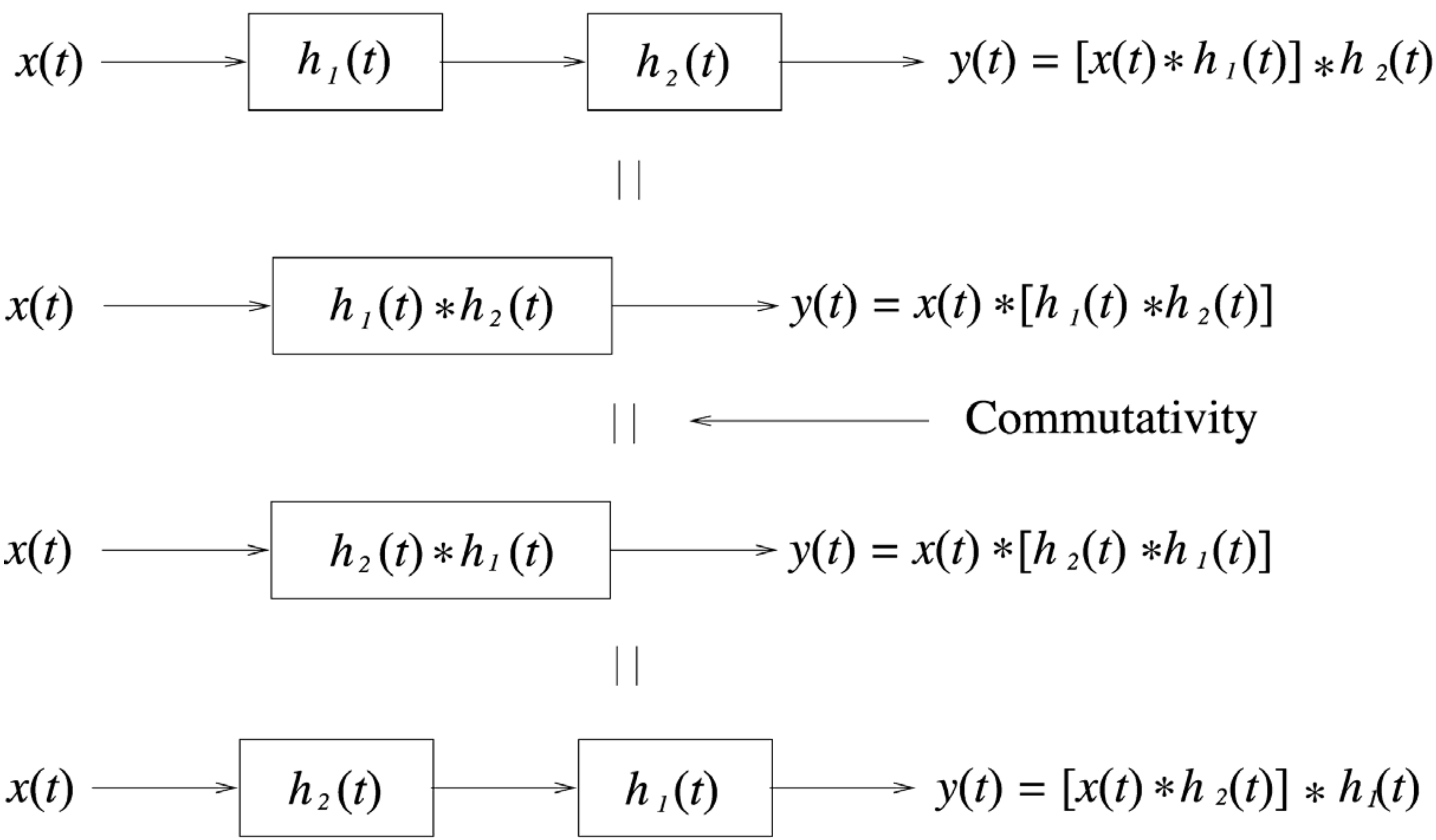
4) Step response:

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

DISTRIBUTIVITY



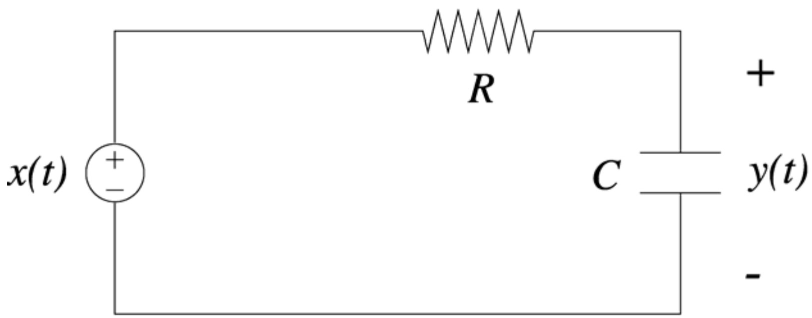
ASSOCIATIVITY



Causality: CT LTI system is causal $\Leftrightarrow h(t) = 0, t < 0$

Stability: CT LTI system is stable $\Leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$

The impulse as an idealized "short" pulse



$$\frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

Consider response from initial rest to pulses of different shapes and durations, but with unit area. As the duration decreases, the responses become similar for different pulse shapes.

The Operational Definition of the Unit Impulse $\delta(t)$

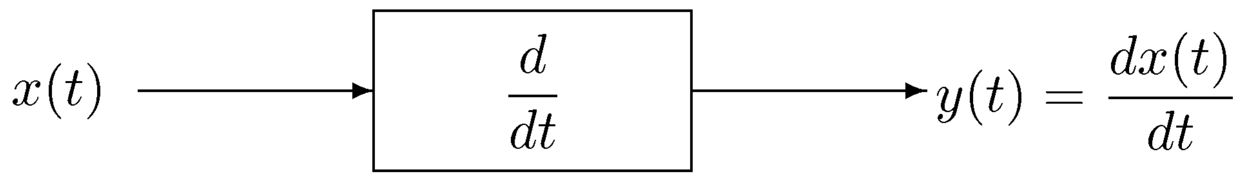
$\delta(t)$ — idealization of a unit-area pulse that is so short that, for any physical systems of interest to us, the system responds only to the area of the pulse and is insensitive to its duration

Operationally: The unit impulse is the signal which when applied to any LTI system results in an output equal to the impulse response of the system. That is,

$$\delta(t) * h(t) = h(t) \qquad \text{for all } h(t)$$

— $\delta(t)$ is defined by what it does under convolution.

The Unit Doublet — Differentiator



Impulse response = unit doublet

$$u_1(t) = \frac{d\delta(t)}{dt}$$

The operational definition of the unit doublet:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

Triplets and beyond!

$n > 0$

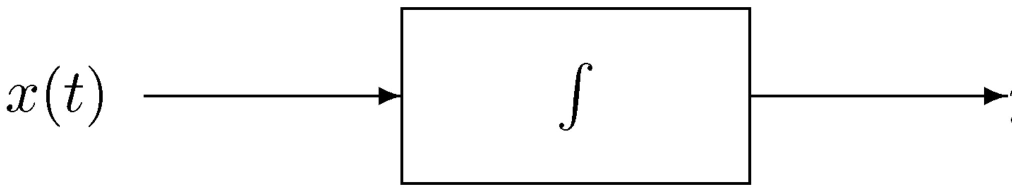
$$u_n(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{\text{n times}}$$

n is number of differentiations

Operational definitions

$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \quad (n > 0)$$

Integrators



$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Impulse response: $u_{-1}(t) \equiv u(t)$

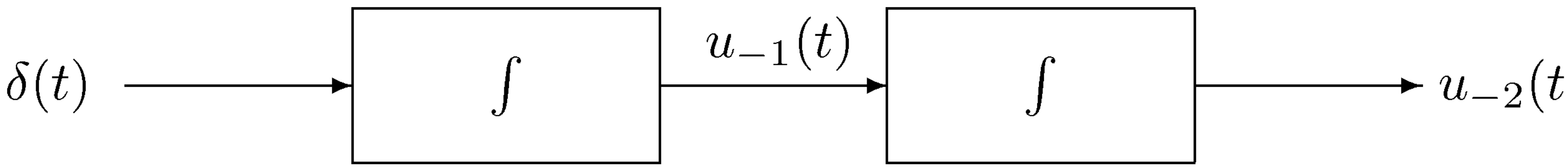
"-1 derivatives" = integral ⇒ I.R. = unit step

Operational definition: $x(t) * u_{-1}(t) = \int_{-\infty}^{t} x(\tau)d\tau$

Cascade of n integrators:

$$u_{-n}(t) = \underbrace{u_{-1}(t) * \cdots * u_{-1}(t)}_{\text{n times}} \quad (n > 0)$$

Integrators (continued)



$$
\begin{aligned}
u_{-2}(t) &= \int_{-\infty}^{t} u_{-1}(\tau)d\tau = \int_{-\infty}^{t} u(\tau)d\tau \\
&= t \cdot u(t) \quad \text{the unit ramp}
\end{aligned}
$$

More generally, for $n > 0$

$$u_{-n}(t) = \frac{t^{n-1}}{(n-1)!}u(t)$$

Notation

Define $$u_0(t) = \delta(t)$$

Then $$u_n(t) * u_m(t) = u_{n+m}(t)$$

n and m can be $\pm$.

E.g. $$u_1(t) * u_{-1}(t) = u_0(t)$$

$$\Downarrow$$

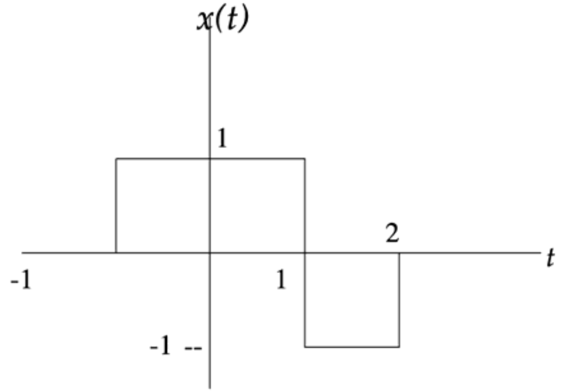
$$\left(\frac{d}{dt}u(t)\right) = \delta(t)$$

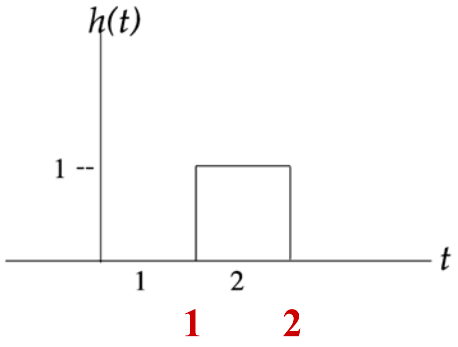
Sometimes Useful Tricks

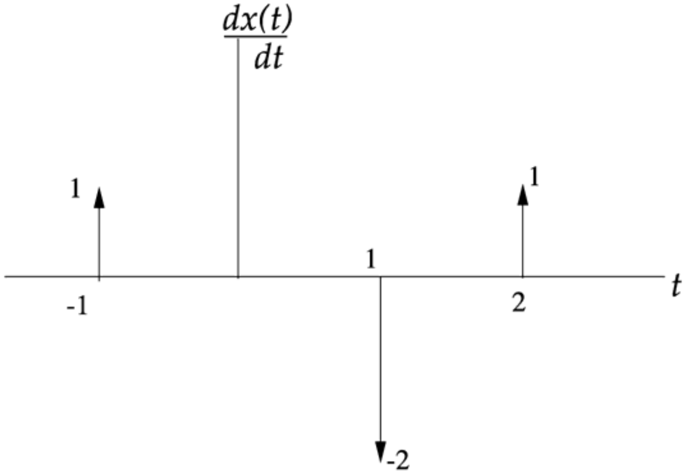
$$
\begin{aligned}
x(t) * h(t) &= x(t) * \delta(t) * h(t) \\
&= x(t) * u_1(t) * u_{-1}(t) * h(t) \\
&= \{[x(t) * u_1(t)] * h(t)\} * u_{-1}(t)
\end{aligned}
$$

Differentiate first, then convolve, then integrate

Example



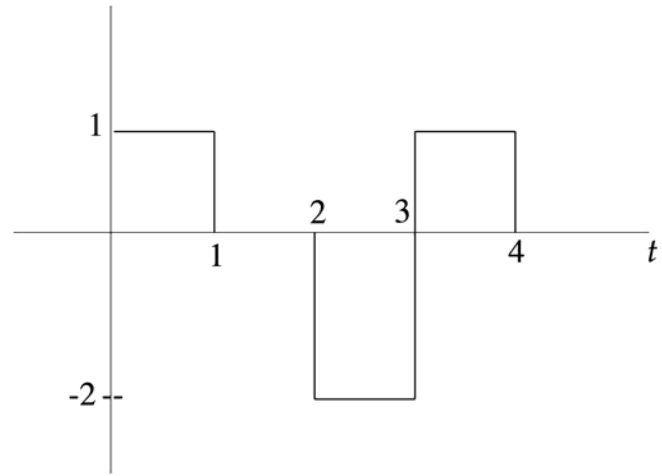




$$\frac{dx(t)}{dt} = \delta(t+1) - 2\delta(t-1) + \delta(t-2)$$

Example (continued)

$$\frac{dx(t)}{dt} * h(t) = h(t+1) - 2h(t-1) + h(t-2)$$



$$x(t) * h(t) = \int_{-\infty}^{t} \left[\frac{dx(\tau)}{d\tau} * h(\tau) \right] d\tau$$