

6.003: Signals and Systems—Fall 2003

PROBLEM SET 3 SOLUTION

Home Study Exercise

(E1) O&W 3.46 (a) and (c)

$x(t)$ and $y(t)$ are continuous-time periodic signals with a period $= T_0$ and Fourier series representations given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad y(t) = \sum_{k=-\infty}^{+\infty} b_k e^{jk\omega_0 t}$$

(a) Show that the Fourier series coefficients of the signal

$z(t) = x(t)y(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$ are given by the discrete convolution $c_k = \sum_{n=-\infty}^{+\infty} a_n b_{k-n}$ (multiplication property).

$$\begin{aligned} z(t) = x(t)y(t) &= \sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t} \sum_{m=-\infty}^{+\infty} b_m e^{jm\omega_0 t} = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} a_n b_m e^{jn\omega_0 t} e^{jm\omega_0 t} \\ &= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} a_n b_m e^{j(n+m)\omega_0 t} \end{aligned}$$

$$\text{Let } k = n + m \rightarrow m = k - n \Rightarrow z(t) = \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a_n b_{k-n} e^{jk\omega_0 t}$$

$$\text{Interchange the summations order } \Rightarrow z(t) = \sum_{k=-\infty}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} a_n b_{k-n} \right) e^{jk\omega_0 t}$$

$$\therefore z(t) = x(t)y(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}, \text{ where } c_k = \sum_{n=-\infty}^{+\infty} a_n b_{k-n}.$$

(c) Suppose that $y(t) = x^*(t)$. Express b_k in terms of a_k , and use the result of part(a) to prove Parseval's relation for periodic signals.

$$y(t) = x^*(t) = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \right)^* = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}$$

$$\text{Let } k = -k \Rightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$$

$$\therefore b_k = a_{-k}^*$$

Now to prove Parseval's relation for period signals we can use the results above as follows:

$$|x(t)|^2 = x(t)x^*(t) = x(t)y(t)$$

$$= \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}, \quad \text{where } c_k = \sum_{n=-\infty}^{+\infty} a_n b_{k-n}, \quad \text{from part(a)}$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} a_n b_{k-n} e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} a_n a_{n-k}^* e^{jk\omega_0 t}, \quad \text{as proven earlier.}$$

A common mistake is to using the substitution $b_{k-n} = a_{k+n}^*$ (i.e. just negating the index n) which would be correct if the relationship was shifting in frequency and *then* taking the conjugate, while the case at hand is taking the conjugate first and *then* shifting in frequency to perform the convolution.

$$P_{ave} = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \frac{1}{T_0} \int_0^{T_0} \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} a_n a_{n-k}^* e^{jk\omega_0 t} dt$$

$$= \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a_n a_{n-k}^* \int_0^{T_0} e^{jk\omega_0 t} dt$$

Notice that the integration has a value of T_0 only when $k = 0$ and zero for all the other values of k , as seen below. This rule, usually referred to as orthogonality, holds for any complex exponential signal, integrated over one period.

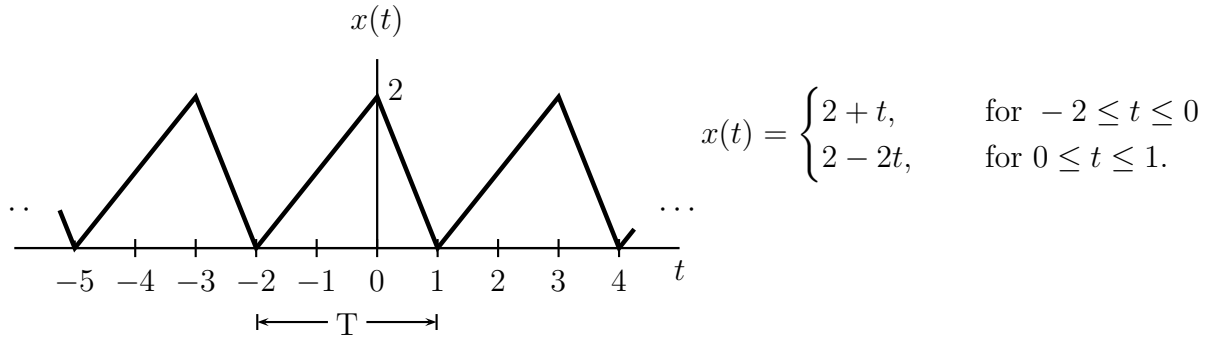
$$\text{For } k \neq 0 : \int_0^{T_0} e^{jk\omega_0 t} dt = \frac{1}{jk\omega_0} e^{jk\omega_0 t} \Big|_0^{T_0} = \frac{1}{jk\omega_0} (e^{jk\omega_0 T_0} - e^{j0})$$

$$= \frac{1}{jk\omega_0} (e^{jk(2\pi)} - e^{j0}) = 0$$

$$\text{For } k = 0 : \int_0^{T_0} e^{jk\omega_0 t} dt = \int_0^{T_0} e^{j(0)\omega_0 t} dt = \int_0^{T_0} (1) dt = T_0.$$

$$\text{Therefore we have: } P_{ave} = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} a_n a_n^* T_0 = \sum_{n=-\infty}^{+\infty} |a_n|^2 = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt.$$

Problem 1 (O&W 3.22 (a) - only the signal in Figure p3.22 (c)) Determine the Fourier series representation for the signal $x(t)$.



$x(t)$ periodic with period $T = 3 \rightarrow \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$

A goal of this problem solution is to show different ways to reaching the same answer. Finding the Fourier series coefficients of a signal using the analysis equation usually requires the most effort, but can be reverted to if everything else fails. Oftentimes, a signal can be dissected into simpler signals that are easier to analyze or can be derived from a simpler signal by integration, differentiation, time shifting, or any combination of the properties of the Fourier series (see Table 3.1, O&W, p.206).

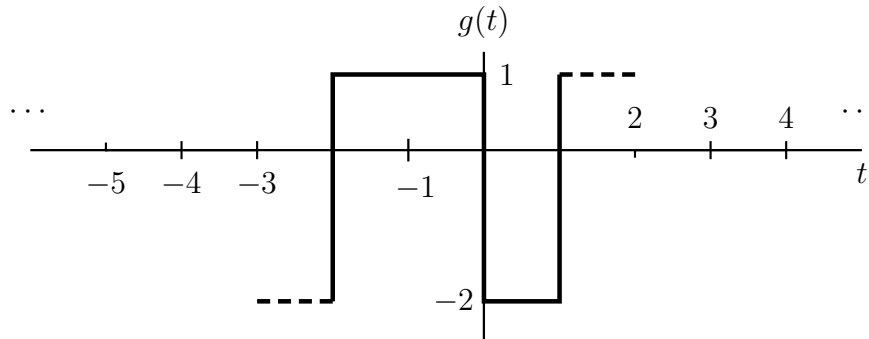
We will start with finding a_0 , which is usually straight-forward and doesn't require much effort, and then explore the different methods for finding $a_{k \neq 0}$:

$$a_0 = \frac{1}{T} \int_T x(t) dt = \frac{1}{3} \text{ (the total area under the curve for one period) } = \frac{1}{3}(2 + 1) = 1.$$

The following are four possible methods to calculate $a_{k \neq 0}$, the Fourier series coefficients of $x(t)$ for $k \neq 0$:

- Method (a): Using the integration property:

Let $g(t) = \frac{dx(t)}{dt} \rightarrow x(t) = \int g(t) dt + p$, where p is the value of $x(t)$ at the beginning of the period, and it equals to zero for the period we selected that starts at $t = -2$. Note that, since we are trying to find $a_{k \neq 0}$, the value of p is not important because it only affects the DC level of $x(t)$ and we have already calculated it by finding a_0 .



Note that $g(t)$ must have a zero DC level, otherwise a ramping signal will be included in $x(t)$ making it non-periodic, and unbounded. By definition, $g(t)$ should have a zero DC level because the derivative operation eliminates it, so this can be used as a double-check.

After finding b_k , the Fourier series coefficients for $g(t)$, we can use the Fourier series properties to find a_k , the Fourier series coefficients for $x(t)$

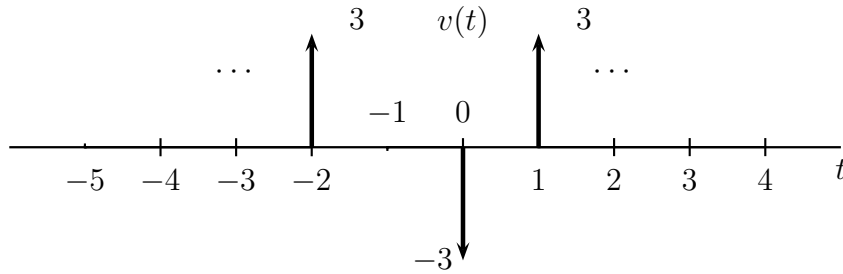
$$\begin{aligned}
 b_k &= \frac{1}{T} \int_T g(t) e^{-jk\omega_0 t} dt = \frac{1}{3} \left(\int_{-2}^0 (1) e^{-jk\omega_0 t} dt + \int_0^1 (-2) e^{-jk\omega_0 t} dt \right) \\
 &= \frac{1}{3} \left(\left. \frac{1}{-jk\omega_0} e^{-jk\omega_0 t} \right|_{-2}^0 - 2 \left. \frac{1}{-jk\omega_0} e^{-jk\omega_0 t} \right|_0^1 \right) = \frac{-1}{3jk\omega_0} (1 - e^{jk\omega_0 2} - 2e^{-jk\omega_0} + 2) \\
 &= \frac{1}{3jk\omega_0} (e^{jk\omega_0 2} + 2e^{-jk\omega_0} - 3). \\
 a_k &= \frac{1}{jk\omega_0} b_k \quad (\text{from the Integration property, Table 3.1, O \&W, p.206}) \\
 &= \frac{1}{jk\omega_0} \frac{1}{3jk\omega_0} (e^{jk\omega_0 2} + 2e^{-jk\omega_0} - 3) = \frac{1}{3k^2\omega_0^2} (3 - 2e^{-jk\omega_0} - e^{jk\omega_0 2}) \\
 &= \frac{1}{k^2\omega_0^2} (1 - e^{jk\omega_0 2}) \quad (\text{remember that } e^{-jk\omega_0} = e^{jk\omega_0 2} \text{ for } T = 3) \\
 &= \frac{1}{k^2\omega_0^2} \left(1 - e^{jk\frac{4\pi}{3}} \right) = \frac{1}{k^2\omega_0^2} \left(1 - e^{-jk\frac{2\pi}{3}} \right).
 \end{aligned}$$

- Method (b): Using the integration property twice:
Let's define $v(t)$ as the following:

$$v(t) = \frac{d^2 x(t)}{dt^2} = \frac{dg(t)}{dt} \Rightarrow x(t) = \int \int v(t) dt dt + p = \int g(t) dt + p$$

Similar to the discussion in Method(a) of the DC level of $g(t)$, $v(t)$ must have a zero DC level. In addition, its limited integration over one period must also have a zero DC level.

We can find $v(t)$ by differentiating $g(t)$. However, in our case, but not always, we can find $v(t)$ directly from $x(t)$ in one step, by placing an impulse at each point of time where the slope of $x(t)$ changes abruptly. The value of that impulse (i.e its area) is the change in slope of $x(t)$ at that point.



To find c_k , the Fourier series coefficients of $v(t)$, let's take the period between -1 and 2, which contains two impulses.

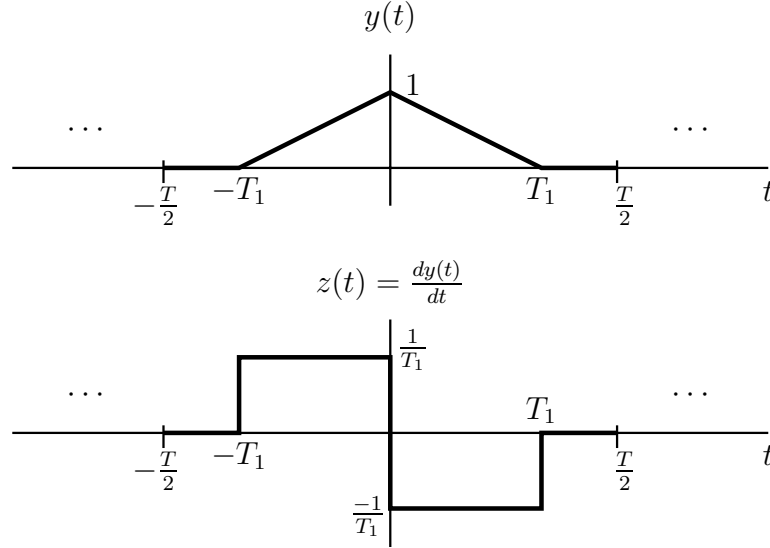
Note that we can also take the period between -2 and 1, but we have to be careful not include the impulses at both -2 and 1. In other words, we can take the period between $-2 + \delta$ and $1 + \delta$ or the period between $-2 - \delta$ and $1 - \delta$.

$$\begin{aligned}
 c_k &= \frac{1}{T} \int_T v(t) e^{-jk\omega_0 t} dt = \frac{1}{3} \left(\int_{-1}^2 [-3\delta(t) + 3\delta(t-1)] e^{-jk\omega_0 t} dt \right) \\
 &= \int_{-1}^2 [-\delta(t) + \delta(t-1)] e^{-jk\omega_0 t} dt \\
 &= -e^{-jk\omega_0(0)} + e^{-jk\omega_0(1)} = e^{-jk\omega_0} - 1.
 \end{aligned}$$

Now to find a_k , we just need to use the integration property two times:

$$\begin{aligned}
 a_k &= \frac{1}{jk\omega_0} \frac{1}{jk\omega_0} c_k \quad (\text{from the Integration property, Table 3.1, O \&W, p.206}) \\
 &= \frac{1}{(jk\omega_0)^2} (e^{-jk\omega_0} - 1) \\
 &= \frac{1}{k^2\omega_0^2} (1 - e^{-jk\omega_0}) \\
 &= \frac{1}{k^2\omega_0^2} \left(1 - e^{-jk\frac{2\pi}{3}} \right), \text{ which is the same answer found in Method(a).}
 \end{aligned}$$

Before exploring the other methods, let's first find the Fourier series for $y(t)$, shown below, which is a periodic triangular function with a period of T . $y(t)$ will be useful for the Method(c):



Let $z(t) = \frac{dy(t)}{dt}$, $z(t) \xleftrightarrow{\mathcal{F}} e_k$, and $y(t) \xleftrightarrow{\mathcal{F}} d_k = \left(\frac{1}{jk\omega_0}\right) e_k$

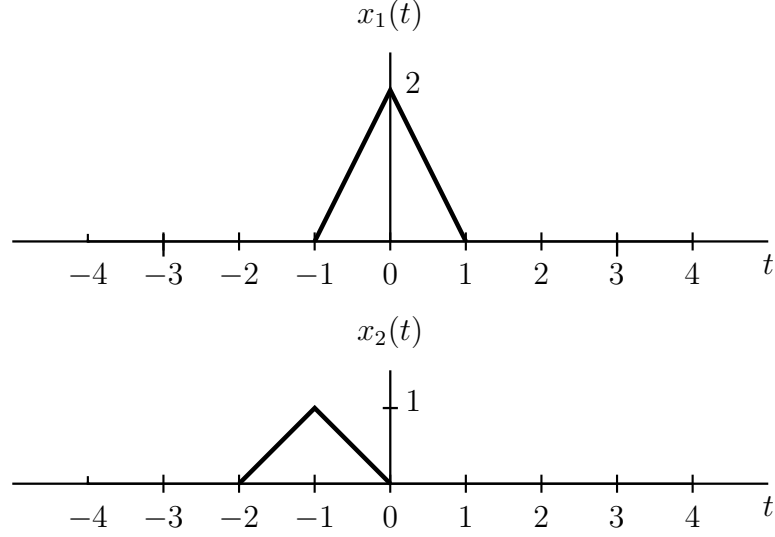
We will find the Fourier series for $z(t)$ and from it, we will find the Fourier series for $y(t)$, as follows:

$$\begin{aligned}
 e_k &= \frac{1}{T} \int_T z(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \left[\int_{-T_1}^0 \left(\frac{1}{T_1}\right) e^{-jk\omega_0 t} dt + \int_0^{T_1} \left(\frac{-1}{T_1}\right) e^{-jk\omega_0 t} dt \right] \\
 &= \frac{1}{TT_1} \frac{1}{(-jk\omega_0)} \left(e^{-jk\omega_0 t} \Big|_{-T_1}^0 - e^{-jk\omega_0 t} \Big|_0^{T_1} \right) = \frac{1}{TT_1} \frac{1}{(-jk\omega_0)} \left(1 - e^{jk\omega_0 T_1} + 1 - e^{-jk\omega_0 T_1} \right) \\
 &= \frac{-1}{TT_1 jk\omega_0} \left[2 - (e^{jk\omega_0 T_1} + e^{-jk\omega_0 T_1}) \right] = \frac{-1}{TT_1 jk\omega_0} \left[2 - 2 \cos(jk\omega_0 T_1) \right].
 \end{aligned}$$

Thus,

$$d_k = e_k \left(\frac{1}{jk\omega_0} \right) = \frac{2 - 2 \cos(k\omega_0 T_1)}{TT_1 k^2 \omega_0^2}.$$

- Method (c): By dissecting the signal into simpler components:
Here, we will dissect $x(t)$ into $x_1(t)$ and $x_2(t)$ which we know their Fourier Series (using the result of d_k above and the time-shifting property).



$x(t) = x_1(t) + x_2(t)$, and let $x_1(t) \xleftrightarrow{\mathcal{F}} b_k$ and $x_2(t) \xleftrightarrow{\mathcal{F}} c_k$

$$\begin{aligned} \therefore a_k &= b_k + c_k = (2) \frac{2 - 2 \cos(k\omega_0(1))}{(3)(1)k^2\omega_0^2} + (1) \frac{2 - 2 \cos(k\omega_0(1))}{(3)(1)k^2\omega_0^2} e^{-jk\omega_0(-1)} \\ &= \frac{2 - 2 \cos k\omega_0}{3k^2\omega_0^2} (2 + e^{jk\omega_0}). \end{aligned}$$

Although this result looks different from those found in the previous methods, further simplification will show that they are identical:

$$\begin{aligned} a_k &= \frac{2 - 2 \cos k\omega_0}{3k^2\omega_0^2} (2 + e^{jk\omega_0}) = \frac{1}{3k^2\omega_0^2} (2 - 2 \cos k\omega_0) (2 + e^{jk\omega_0}) \\ &= \frac{1}{3k^2\omega_0^2} (2 - e^{jk\omega_0} - e^{-jk\omega_0}) (2 + e^{jk\omega_0}) \\ &= \frac{1}{3k^2\omega_0^2} (4 + 2e^{jk\omega_0} - 2e^{jk\omega_0} - e^{jk\omega_0 2} - 2e^{-jk\omega_0} - e^0) \\ &= \frac{1}{3k^2\omega_0^2} (4 - e^{jk\omega_0 2} - 2e^{-jk\omega_0} - 1) = \frac{1}{3k^2\omega_0^2} (3 - e^{jk\omega_0 2} - 2e^{-jk\omega_0}) \\ &= \frac{1}{k^2\omega_0^2} (1 - e^{jk\omega_0 2}) \quad (\text{remember that } e^{-jk\omega_0} = e^{jk\omega_0 2} \text{ for } T = 3) \\ &= \frac{1}{k^2\omega_0^2} \left(1 - e^{jk\frac{4\pi}{3}}\right), \text{ which is the same answer found in previous methods.} \end{aligned}$$

- Method (d): using the analysis equation:

In the process of evaluating the analysis equation, the following integral will save us a lot of derivation steps:

$$\int te^{at} dt = \left(\frac{t}{a} - \frac{1}{a^2} \right) e^{at} \quad , \text{ for any } a \neq 0$$

$$\begin{aligned}
a_k &= \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt = \frac{1}{3} \int_{-2}^1 x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{3} \left[\int_{-2}^0 (2+t) e^{-jk\omega_0 t} dt + \int_0^1 (2-2t) e^{-jk\omega_0 t} dt \right] \\
&= \frac{1}{3} \left[2 \int_{-2}^1 e^{-jk\omega_0 t} dt + \int_{-2}^0 t e^{-jk\omega_0 t} dt - 2 \int_0^1 t e^{-jk\omega_0 t} dt \right] \\
&= \frac{1}{3} \left[2 \left. \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right|_{-2}^1 + \left(\frac{1}{k^2\omega_0^2} - \frac{t}{jk\omega_0} \right) e^{-jk\omega_0 t} \Big|_{-2}^0 - 2 \left(\frac{1}{k^2\omega_0^2} - \frac{t}{jk\omega_0} \right) e^{-jk\omega_0 t} \Big|_0^1 \right] \\
&= \frac{1}{3} \left\{ \frac{-2}{jk\omega_0} (e^{-jk\omega_0(1)} - e^{-jk\omega_0(-2)}) + \frac{1}{k^2\omega_0^2} - \left(\frac{1}{k^2\omega_0^2} - \frac{(-2)}{jk\omega_0} \right) e^{-jk\omega_0(-2)} \right. \\
&\quad \left. - 2 \left[\left(\frac{1}{k^2\omega_0^2} - \frac{(1)}{jk\omega_0} \right) e^{-jk\omega_0(1)} - \frac{1}{k^2\omega_0^2} \right] \right\} \\
&= \frac{1}{3} \left(\frac{-2}{jk\omega_0} e^{-jk\omega_0} + \frac{2}{jk\omega_0} e^{jk\omega_0 2} + \frac{1}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2} e^{jk\omega_0 2} - \frac{2}{jk\omega_0} e^{jk\omega_0 2} \right. \\
&\quad \left. - \frac{2}{k^2\omega_0^2} e^{-jk\omega_0} + \frac{2}{jk\omega_0} e^{-jk\omega_0} + \frac{2}{k^2\omega_0^2} \right) \\
&= \frac{1}{3} \left(\frac{-2}{jk\omega_0} e^{-jk\omega_0} - \frac{2}{k^2\omega_0^2} e^{-jk\omega_0} + \frac{2}{jk\omega_0} e^{-jk\omega_0} + \frac{1}{k^2\omega_0^2} + \frac{2}{k^2\omega_0^2} \right. \\
&\quad \left. + \frac{2}{jk\omega_0} e^{jk\omega_0 2} - \frac{1}{k^2\omega_0^2} e^{jk\omega_0 2} - \frac{2}{jk\omega_0} e^{jk\omega_0 2} \right) \\
&= \frac{1}{3} \left(-\frac{2}{k^2\omega_0^2} e^{-jk\omega_0} + \frac{3}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2} e^{jk\omega_0 2} \right) \\
&= \frac{1}{3k^2\omega_0^2} (3 - 2e^{-jk\omega_0} - e^{jk\omega_0 2}) \\
&= \frac{1}{k^2\omega_0^2} (1 - e^{jk\omega_0 2}) \quad (\text{remember that } e^{-jk\omega_0} = e^{jk\omega_0 2} \text{ for } T = 3) \\
&= \frac{1}{k^2\omega_0^2} (1 - e^{jk\frac{4\pi}{3}}), \text{ which is the same answer found in previous methods.} \\
&= \frac{9}{4k^2\pi^2} (1 - e^{jk\frac{4\pi}{3}}).
\end{aligned}$$

Problem 2 O & W 3.23 (a)

Given a_k , the Fourier series coefficients of a periodic continuous time signal with period 4, determine the signal $x(t)$.

The Fourier series coefficients a_k are given as follows:

$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \frac{\sin k\pi/4}{k\pi}, & \text{otherwise.} \end{cases}$$

Here are some of the facts we know about $x(t)$:

- $a_0 = 0 \rightarrow$ no DC component in $x(t)$
- $T = 4 \rightarrow \omega_0 = 2\pi/4 = \pi/2$
-

$$\begin{aligned} a_{-k} &= (j)^{-k} \frac{\sin(-k\pi/4)}{-k\pi} = \left(\frac{1}{j}\right)^k \frac{-\sin(k\pi/4)}{-k\pi} \\ &= (-j)^k \frac{\sin(k\pi/4)}{k\pi} = a_k^*. \end{aligned}$$

Thus $x(t)$ is a real signal (O&W, Section 3.5.6, p.204).

Noting that $j = e^{j\pi/2} \rightarrow (j)^k = (e^{j\pi/2})^k = e^{jk\pi/2} = e^{jk\omega_0} = e^{-jk\omega_0(-1)}$, we can consider $x(t)$ to be a time-shifted version of another signal $y(t)$ such that:

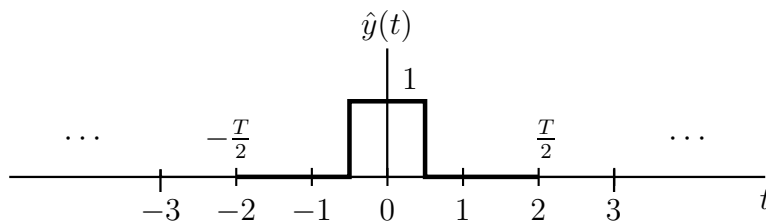
$$x(t) = y(t+1), \text{ where } y(t) \xleftrightarrow{\mathcal{F}} b_0 = 0, b_{k \neq 0} = \frac{\sin k\pi/4}{k\pi} \text{ and } a_k = b_k e^{jk\omega_0(1)}$$

By backtracking the derivation equation of b_k , we can find the signal $\hat{y}(t)$ which has the same b_k but can have a different DC level (i.e. b_0):

$$\begin{aligned} \hat{b}_{k \neq 0} = b_{k \neq 0} &= \frac{\sin k\pi/4}{k\pi} = \frac{1}{k\pi} \left(\frac{e^{jk\pi/4} - e^{-jk\pi/4}}{2j} \right) \\ &= \frac{1}{(4)jk(\frac{\pi}{2})} (e^{jk\pi/4} - e^{-jk\pi/4}) \\ &= \frac{1}{T} \frac{1}{jk\omega_0} \left(e^{jk\omega_0(\frac{1}{2})} - e^{jk\omega_0(-\frac{1}{2})} \right) = \frac{1}{T} \int_{-\frac{1}{2}}^{\frac{1}{2}} (1) e^{jk\omega_0 t} dt. \end{aligned}$$

The integration above suggests that

$$\hat{y}(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{elsewhere in the same period } T=4. \end{cases}$$



Note that the same conclusion can be reached by noticing that $\hat{y}(t)$ is the same signal in Example 3.5 (O& W, p.193) with $T_1 = \frac{1}{2}$ and $T = 4$.

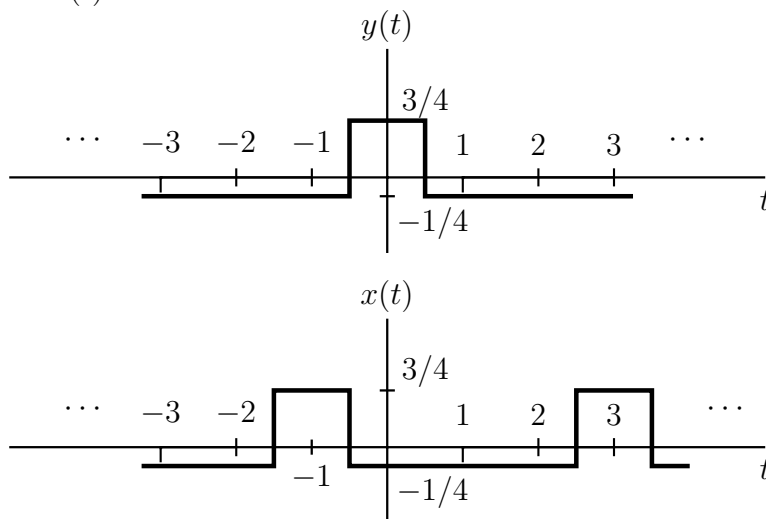
To find $y(t)$, which has $b_0 = 0$, we first calculate \hat{b}_0 and then subtract it from $\hat{y}(t)$:

$$\hat{b}_0 = \frac{1}{T} \int_T \hat{y}(t) dt = \frac{1}{4} \int_{-1/2}^{1/2} (1) dt = \frac{1}{4}$$

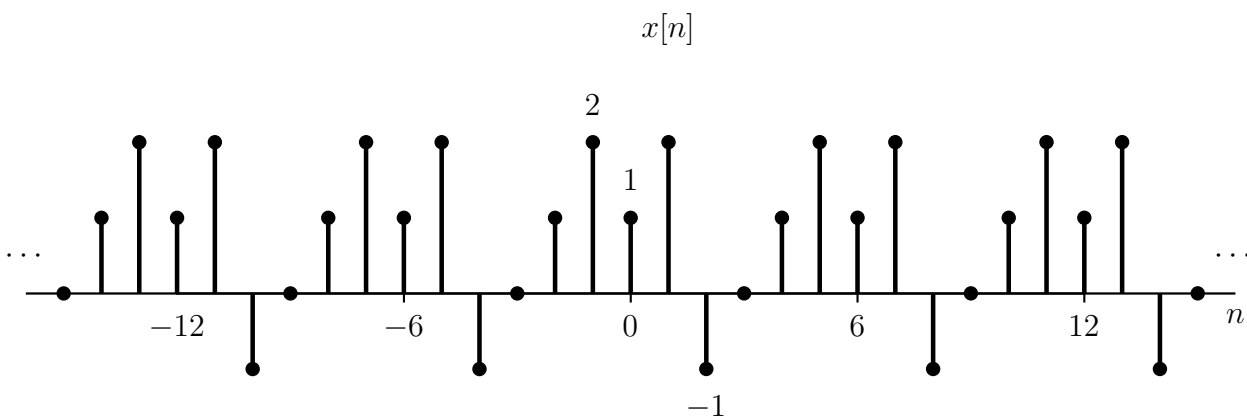
$$\rightarrow y(t) = \hat{y}(t) - \frac{1}{4} \Rightarrow y(t) = \begin{cases} \frac{3}{4}, & -\frac{1}{2} < t < \frac{1}{2} \\ -\frac{1}{4}, & \frac{1}{2} < |t| < 2. \end{cases}$$

$$\rightarrow x(t) = y(t+1) = \begin{cases} \frac{3}{4}, & -1.5 < t < -0.5 \\ -\frac{1}{4}, & -0.5 < t < 2.5 \end{cases}$$

Sketches of $y(t)$ and $x(t)$ are shown below:



Problem 3 Determine the Fourier series coefficients for the periodic signal $x[n]$ depicted below. Plot the magnitude and phase of these coefficients.



Fundamental period $N = 6 \rightarrow \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$.

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{6} \sum_{n=0}^5 x[n]e^{-jk\omega_0 n} = \frac{1}{6} \sum_{n=-3}^2 x[n]e^{-jk\omega_0 n}$$

Notice that the last two expressions will give the same result, but the latter would take advantage of the symmetry of some of the samples to combine them into sinusoids.

$$\begin{aligned} a_k &= \frac{1}{6} [(0)e^{-jk\omega_0(-3)} + (1)e^{-jk\omega_0(-2)} + (2)e^{-jk\omega_0(-1)} + (1)e^{-jk\omega_0(0)} + \\ &\quad + (2)e^{-jk\omega_0(1)} + (-1)e^{-jk\omega_0(2)}] \\ &= \frac{1}{6} [e^{-jk\omega_0(-2)} - e^{-jk\omega_0(2)} + 2e^{-jk\omega_0(-1)} + 2e^{-jk\omega_0(1)} + 1] \\ &= \frac{1}{6} [(2j) \sin k\omega_0 2 + 2(2) \cos k\omega_0 + 1] \\ &= \frac{1}{6} + \frac{2}{3} \cos k\omega_0 + \frac{j}{3} \sin k\omega_0 2 \\ \therefore a_k &= \frac{1}{6} + \frac{2}{3} \cos \left(k \frac{\pi}{3} \right) + j \frac{1}{3} \sin \left(k \frac{2\pi}{3} \right) \\ &= \frac{1}{6} \left[1 + 4 \cos \left(k \frac{\pi}{3} \right) + j 2 \sin \left(k \frac{2\pi}{3} \right) \right] \end{aligned}$$

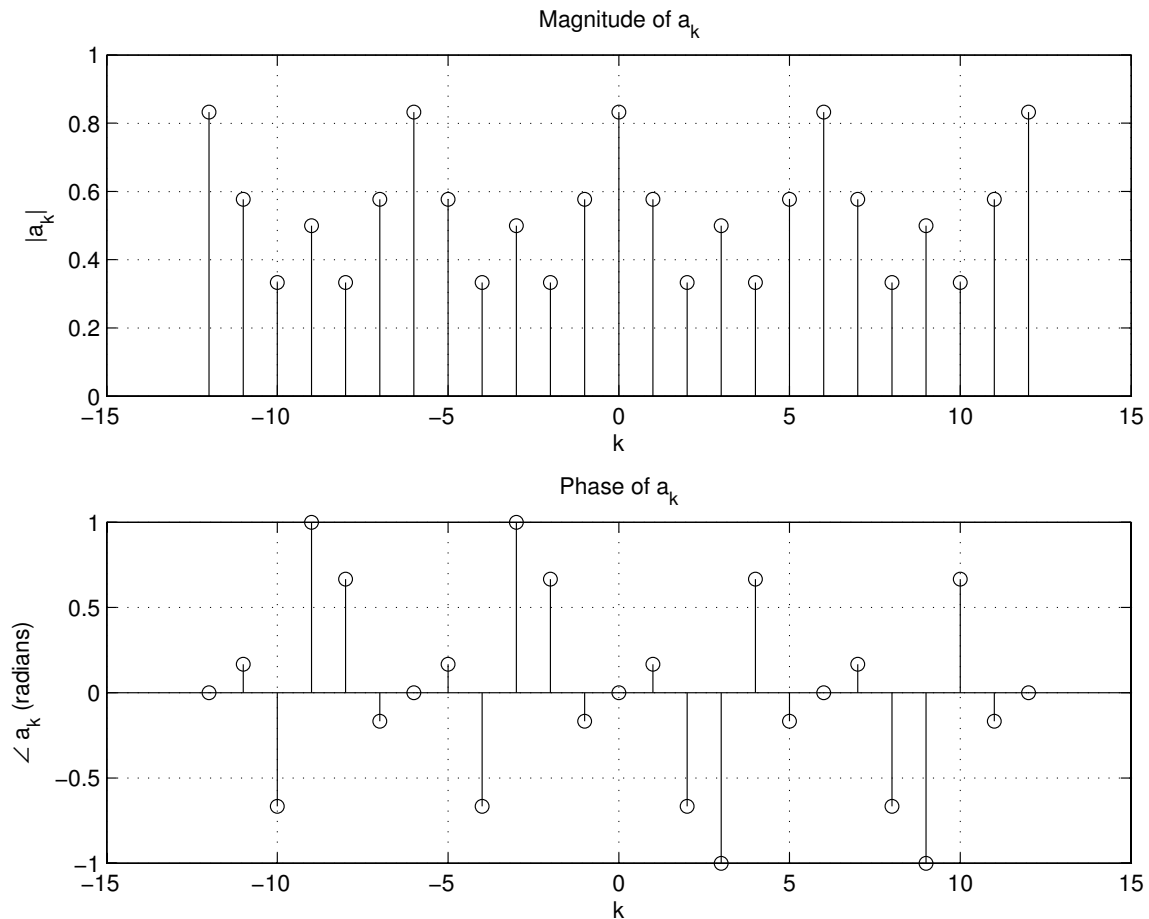
Here are the values of a_k for one period of six consecutive points (from $k=-2$ to $k=3$):

$$\begin{aligned}
 a_{-2} &= \frac{1}{6} \left[1 + 4 \cos \left((-2) \frac{\pi}{3} \right) + 2j \sin \left((-2) \frac{2\pi}{3} \right) \right] \\
 &= \frac{1}{6} \left[1 + 4 \cos \left(\frac{2\pi}{3} \right) - 2j \sin \left(\frac{4\pi}{3} \right) \right] = \frac{1}{6} \left[1 + 4 \left(\frac{-1}{2} \right) - 2j \left(\frac{-\sqrt{3}}{2} \right) \right] \\
 &= \frac{1}{6} (-1 + j\sqrt{3}) = -0.1667 + j0.2887 \\
 &= \frac{\sqrt{(-1)^2 + (\sqrt{3})^2}}{6} e^{j \operatorname{atan}(\sqrt{3}, -1)} = \frac{1}{3} e^{j \frac{2\pi}{3}} \rightarrow |a_{-2}| = \frac{1}{3}, \angle a_{-2} = \frac{2\pi}{3}.
 \end{aligned}$$

Similarly, for the magnitude and phase of a_k for $k = -1 \rightarrow 3$ which are summarized in the table below:

k	a_k	$ a_k $	$\angle a_k$
-2	$-0.1667 + j 0.2887$	$1/3$	$2\pi/3$
-1	$0.5000 - j 0.2887$	$1/\sqrt{3}$	$-\pi/6$
0	0.8333	$5/6$	0
1	$0.5000 + j 0.2887$	$1/\sqrt{3}$	$\pi/6$
2	$-0.1667 - j 0.2887$	$1/3$	$-2\pi/3$
3	-0.5000	$3/2$	$-\pi$

The magnitude and phase of the Fourier series coefficients were plotted below, using MATLAB:



For your reference, the MATLAB code used to compute and plot the magnitude and phase of the Fourier series coefficients is shown below:

```
MATLAB Code:
A=inline('1/6 +2/3*cos(k*pi/3)+j/3*sin(k*2*pi/3)');
k=-12:12;a=A(k);am=abs(a);ap=angle(a); subplot(2,1,1);stem(k,am);grid
on;xlabel('k');ylabel('|a_k|');title('Magnitude of a_k');
subplot(2,1,2);stem(k,ap/pi);grid on; xlabel('k');ylabel('\angle a_k
(radians)');title('Phase of a_k');
```

MATLAB tip: you can use TEX expressions in the text of figures.

Problem 4 O & W 3.29 (a)

Given a_k , the Fourier series coefficients of a periodic discrete time signal with period 8, determine the signal $x[n]$.

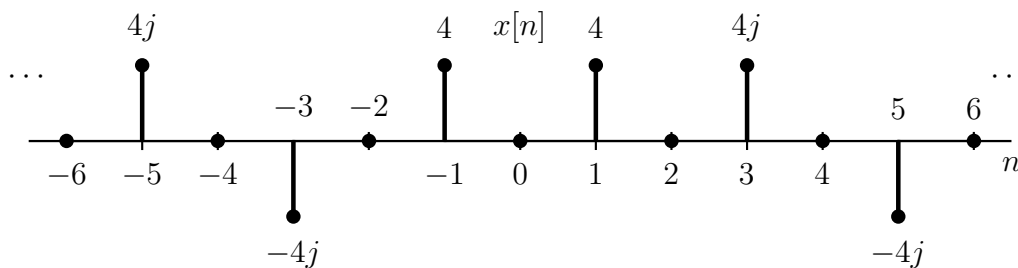
The Fourier coefficients are given as follows:

$$a_k = \cos \frac{k\pi}{4} + \sin \frac{3k\pi}{4}.$$

$N=8 \rightarrow$ there are only 8 samples to compute in $x[n]$, some of which can have a zero value, $\omega_0 = 2\pi/8 = \pi/4$.

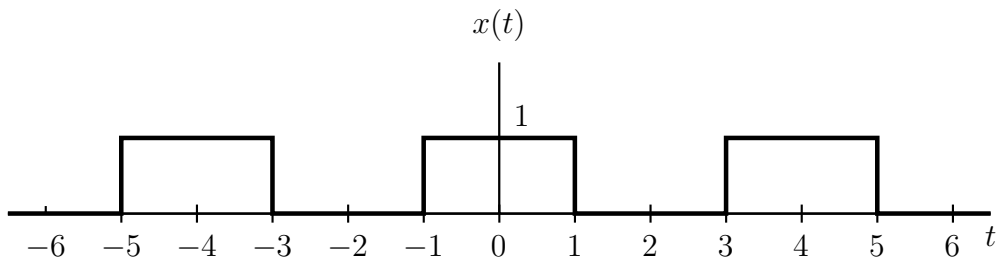
$$\begin{aligned} a_k &= \cos \frac{k\pi}{4} + \sin \frac{3k\pi}{4} = \cos k\omega_0 + \sin 3k\omega_0 \\ &= \frac{1}{2} e^{jk\omega_0} + \frac{1}{2} e^{-jk\omega_0} + \frac{1}{2j} e^{j3k\omega_0} - \frac{1}{2j} e^{-j3k\omega_0} \\ &= \frac{1}{8} \left[4 e^{-jk\omega_0(-1)} + 4 e^{-jk\omega_0(1)} + \frac{4}{j} e^{-jk(-3)\omega_0} - \frac{4}{j} e^{-jk(3)\omega_0} \right] \\ &= \frac{1}{8} \left[(4) e^{-jk\omega_0(-1)} + (4) e^{-jk\omega_0(1)} + (-4j) e^{-jk(-3)\omega_0} + (4j) e^{-jk(3)\omega_0} \right] \\ &= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{8} \sum_{n=-3}^4 x[n] e^{-jk\omega_0 n} \end{aligned}$$

By matching the expressions of $a_k \rightarrow x[n] = \begin{cases} -4j, & n = -3 \\ 4j, & n = 3 \\ 4, & n = \pm 1 \\ 0, & n = 0, \pm 2, 4 \end{cases}$



Problem 5 Consider the following CT periodic signals, $x(t)$, $y(t)$, and $z(t)$.

- (a) Determine the fundamental frequency, period, and Fourier series coefficients, a_k , for $x(t)$.

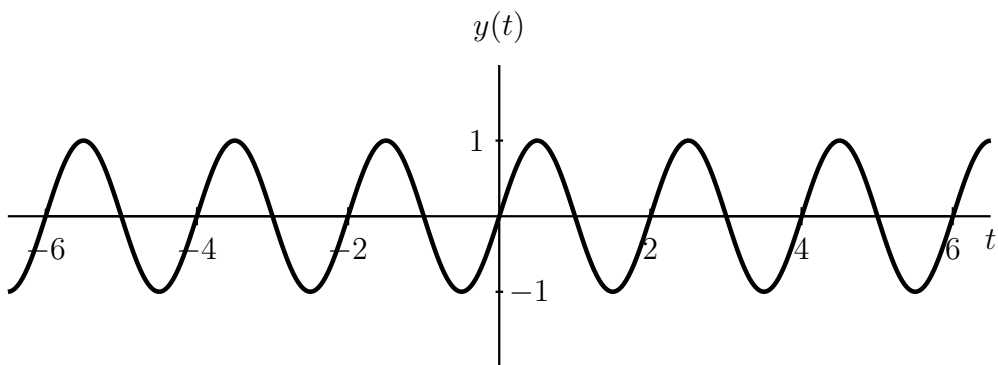


Fundamental period of $x(t) = T = 4 \rightarrow \omega_0 = 2\pi/4 = \pi/2$.

$$\begin{aligned}
 a_0 &= \frac{1}{T} \int_T x(t) dt = \frac{1}{4} \int_{-1}^1 x(t) dt = \frac{2}{4} = \frac{1}{2}. \\
 a_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{4} \int_{-1}^1 (1) e^{-jk\omega_0 t} dt = \frac{1}{-jk\omega_0 4} e^{-jk\omega_0 t} \Big|_{-1}^1 \\
 &= \frac{1}{jk2\pi} (e^{jk\omega_0} - e^{-jk\omega_0}) = \frac{\sin(k\omega_0)}{k\pi} = \frac{\sin(k\frac{\pi}{2})}{k\pi}.
 \end{aligned}$$

The same result can also be found directly using Example 3.5 (O & W, P.193).

- (b) Determine the fundamental frequency, period, and Fourier series coefficients, b_k , for $y(t)$.



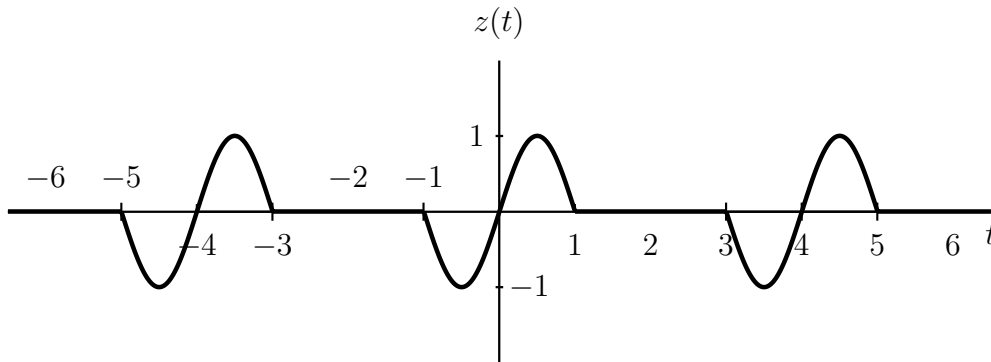
Fundamental period of $y(t) = \hat{T} = 2 \rightarrow \hat{\omega}_0 = 2\pi/2 = \pi$.

$$b_0 = \frac{1}{T} \int_T y(t) dt = 0 \quad (\because y(t) \text{ has no DC component}).$$

$$y(t) = \sin \hat{\omega}_0 t = \frac{e^{j\hat{\omega}_0 t} - e^{-j\hat{\omega}_0 t}}{2j} = \frac{-j}{2} e^{j(1)\hat{\omega}_0 t} - \frac{-j}{2} e^{j(-1)\hat{\omega}_0 t} = \sum_{k=-\infty}^{+\infty} b_k e^{jk\hat{\omega}_0 t}$$

$$\rightarrow b_k = \begin{cases} \frac{-j}{2}, & k = 1 \\ \frac{j}{2}, & k = -1 \\ 0, & \text{otherwise} \end{cases}$$

- (c) Determine the fundamental frequency and period for $z(t)$. Also, using the results of parts (a) and (b), determine the Fourier series coefficients, c_k for $z(t)$.



Fundamental period of $z(t)$ = Fundamental period of $x(t)$ = $T = 4 \rightarrow \omega_0 = 2\pi/4 = \frac{\pi}{2}$.

$$c_0 = \frac{1}{T} \int_T z(t) dt = 0 \quad (\because z(t) \text{ has no DC component}).$$

Noticing that $z(t) = x(t)y(t)$, we can find c_k using the multiplication property. However, the fundamental frequencies of $x(t)$ and $y(t)$ must be identical in order for the Fourier coefficients to match (i.e. to represent the same frequencies). The fundamental period of $y(t)$ is 2, but if we define it to be 4, then we only need to scale the frequency components accordingly to keep the value of $k\omega_0$ constant. In our case, for $y(t)$: $\omega_0 = \frac{\pi}{2} = \hat{\omega}_0/2$

$$b'_k = \begin{cases} \frac{-j}{2}, & k = 2 \\ \frac{j}{2}, & k = -2 \\ 0, & \text{otherwise} \end{cases}$$

Using the multiplication property:

$$c_k = \sum_{n=-\infty}^{+\infty} a_n b'_{k-n}, \text{ which looks like the discrete-time convolution, in frequency.}$$

Note that $a_n b'_{k-n} \neq 0$ only when $k - n = \pm 2$

$$\rightarrow c_k = a_{k-2} b'_2 + a_{k+2} b'_{-2} = a_{k-2} \frac{-j}{2} + a_{k+2} \frac{j}{2} = \frac{j}{2} (a_{k+2} - a_{k-2})$$

$$a_{k+2} = \begin{cases} \frac{\sin(k+2)\frac{\pi}{2}}{(k+2)\pi}, & k \neq -2 \\ \frac{1}{2}, & k = -2 \end{cases} = \begin{cases} \frac{-\sin(k\frac{\pi}{2})}{(k+2)\pi}, & k \neq -2 \\ \frac{1}{2}, & k = -2 \end{cases}$$

$$a_{k-2} = \begin{cases} \frac{\sin(k-2)\frac{\pi}{2}}{(k-2)\pi}, & k \neq 2 \\ \frac{1}{2}, & k = 2 \end{cases} = \begin{cases} \frac{-\sin(k\frac{\pi}{2})}{(k-2)\pi}, & k \neq 2 \\ \frac{1}{2}, & k = 2 \end{cases}$$

Problem 6 Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficients a_k . Derive the Fourier series coefficients of each of the following signals in terms of a_k :

(a) $\mathcal{O}d\{x(t - T/2)\}$

$$\begin{aligned} x(t - T/2) \longleftrightarrow b_k &= a_k e^{-jk\omega_0 \frac{T}{2}} \quad (\text{Time Shifting Property}) \\ &= a_k e^{-jk\pi} = a_k (e^{-j\pi})^k \\ &= a_k (-1)^k \end{aligned}$$

If we assume that $x(t)$ is real, then:

$$\begin{aligned} \mathcal{O}d\{x(t - T/2)\} \longleftrightarrow c_k &= j \Im\{b_k\} \quad (\text{Even-Odd Decomposition of Real Signals} \\ &\quad \text{Propriety, Table 3.1, O \& W, p. 206}) \\ &= j \Im\{a_k (-1)^k\} = (-1)^k j \Im\{a_k\}. \end{aligned}$$

However, the question didn't specify $x(t)$ to be real, so assuming that $x(t)$ is complex, we will just use the general formula for finding the Odd part of a signal:

$$\mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)] \quad (\text{O \& W, Sec. 1.2.3, and specifically eq.(1.19), p.14})$$

$$\begin{aligned} \mathcal{O}d\{x(t - T/2)\} = \frac{1}{2}[x(t - T/2) - x(-t - T/2)] \longleftrightarrow d_k &= \frac{1}{2}[a_k (-1)^k - a_{-k} (-1)^{-k}] \\ &= \frac{1}{2}(-1)^k (a_k - a_{-k}). \end{aligned}$$

Note that for real $x(t)$: $a_{-k} = a_k^* \rightarrow a_k - a_{-k} = 2j \Im\{a_k\} \rightarrow c_k = d_k$.

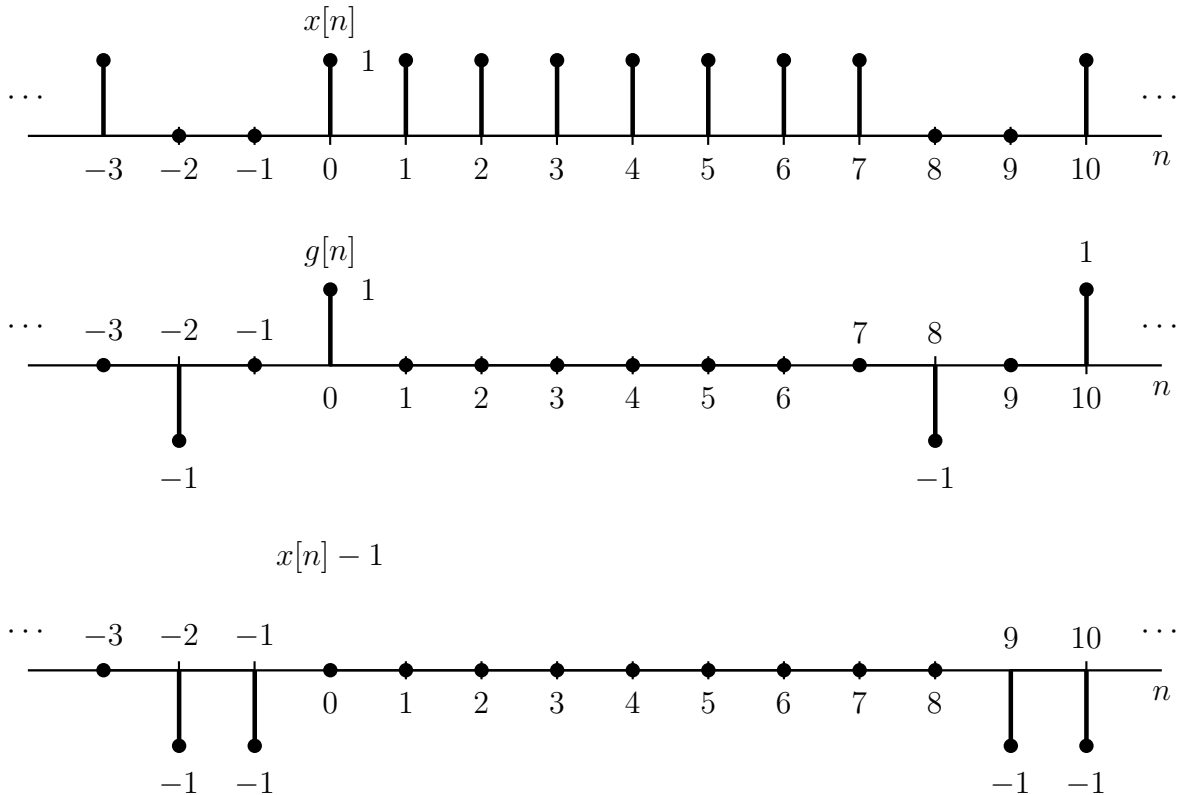
(b) $x(T/4 - t)$

$$\begin{aligned} x(-t) \longleftrightarrow c_k &= a_{-k} \quad (\text{Time-Reversal Property}) \\ x(T/4 - t) \longleftrightarrow d_k &= c_k e^{-jk\omega_0 T/4} \quad (\text{Time-Shift in the positive time direction, i.e. delay}) \\ &= c_k e^{-jk\frac{\pi}{2}} = c_k (-j)^k = a_{-k} (-j)^k. \end{aligned}$$

Problem 7 O & W 3.31 (also determine a_0)

Let $x[n] = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0, & 8 \leq n \leq 9 \end{cases}$, $x[n]$: periodic, $N = 10$, Fourier series coefficients: a_k .

Also, let $g[n] = x[n] - x[n - 1]$.



Fundamental Period = $N = 10 \rightarrow \omega_0 = \frac{\pi}{5}$.

$$a_0 = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] = \frac{1}{10} \sum_{n=0}^9 x[n] = \frac{1}{10} [1(8) + 0(2)] = 8/10 = 4/5.$$

(a) Show that $g[n]$ has a fundamental period of 10.

$$\begin{aligned} g[n + N] &= x[n + N] - x[n + N - 1] \\ \because x[n + N] &= x[n] \rightarrow g[n + N] = x[n] - x[n - 1] = g[n] \\ &\rightarrow g[n] \text{ has a fundamental period of } N = 10. \end{aligned}$$

(b) Determine the Fourier series coefficients of $g[n]$.

$$g[n] = x[n] - x[n-1] = \begin{cases} 1, & n = 0 \\ 0, & 1 \leq n \leq 7 \\ -1, & n = 8. \end{cases}$$

$$\begin{aligned} b_k &= \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk\omega_0 n} \\ &= \frac{1}{10} \sum_{n=-2}^7 g[n] e^{-jk\omega_0 n}, \text{ the limits were chosen to use the non-zeros near the origin} \\ &= \frac{1}{10} [(-1)e^{-jk\omega_0(-2)} + (1)e^{-jk\omega_0(0)}] = \frac{1}{10} [1 - e^{jk\omega_0 2}] = \frac{1}{10} e^{jk\omega_0} [e^{-jk\omega_0} - e^{jk\omega_0}] \\ &= \frac{-j2}{10} e^{jk\omega_0} \left[\frac{e^{jk\omega_0} - e^{-jk\omega_0}}{2j} \right] = \frac{-j}{5} e^{jk\omega_0} \sin k\omega_0 = \frac{-j}{5} e^{jk\frac{\pi}{5}} \sin k\frac{\pi}{5} \end{aligned}$$

(c) Using the Fourier series coefficients of $g[n]$ and the First-Difference property in Table 3.2, determine a_k for $k \neq 0$.

From Table 3.2 (O & W, p. 221): $x[n] - x[n-1] \longleftrightarrow (1 - e^{-jk(2\pi/N)}) a_k = b_k$

$$\begin{aligned} &\rightarrow b_k = \frac{1}{10} (1 - e^{jk\omega_0 2}) = (1 - e^{-jk\omega_0}) a_k \\ a_k &= \frac{1}{10} \frac{1 - e^{jk\omega_0 2}}{1 - e^{-jk\omega_0}} = \frac{1}{10} \frac{(1 - e^{jk\omega_0})(1 + e^{jk\omega_0})}{e^{-jk\omega_0}(e^{jk\omega_0} - 1)} = \frac{-1}{10} \frac{(1 + e^{jk\omega_0})}{e^{-jk\omega_0}} \\ &= \frac{-1}{10} \frac{e^{jk\omega_0 \frac{1}{2}}}{e^{-jk\omega_0}} (e^{-jk\omega_0 \frac{1}{2}} + e^{jk\omega_0 \frac{1}{2}}) = \frac{-1}{5} e^{jk\omega_0 \frac{3}{2}} \cos \left(k\omega_0 \frac{1}{2} \right) \\ &= \frac{-1}{5} e^{jk\frac{3\pi}{10}} \cos \left(k\frac{\pi}{10} \right). \end{aligned}$$

Let's double check the result, and at the same time use another route to find a_k :

Note that $x[n] - 1$ would have the same a_k (only a_0 changes with a change in the DC level of a signal).

$$\begin{aligned} x[n] - 1 &= \begin{cases} 0, & 0 \leq n \leq 7 \\ -1, & 8 \leq n \leq 9 \end{cases} = \begin{cases} 0, & 0 \leq n \leq 7 \\ -1, & -2 \leq n \leq -1 \end{cases} \\ a_k &= \frac{1}{N} \sum_{n=\langle N \rangle} (x[n] - 1) e^{-jk\omega_0 n} = \frac{1}{10} [(-1)e^{-jk\omega_0(-2)} + (-1)e^{-jk\omega_0(-1)}] = \frac{-1}{10} [e^{jk\omega_0 2} + e^{jk\omega_0}] \\ &= \frac{-1}{10} e^{jk\omega_0(3/2)} [e^{jk\omega_0(1/2)} + e^{-jk\omega_0(1/2)}] = \frac{-1}{5} e^{jk\omega_0(3/2)} \cos(k\omega_0 \frac{1}{2}) = \frac{-1}{5} e^{jk\frac{3\pi}{10}} \cos \left(k\frac{\pi}{10} \right). \end{aligned}$$

Problem 8 O & W 3.51

$x[n]$: periodic signal with period $N = 8$ and Fourier series coefficients $a_k = -a_{k-4}$.

$y[n]$: periodic signal with period $N = 8$ and Fourier series coefficients b_k ,

$$y[n] = \left(\frac{1 + (-1)^n}{2} \right) x[n - 1]$$

Find a function $f[k]$ such that $b_k = f[k]a_k$.

$$y[n] = \left(\frac{1 + (-1)^n}{2} \right) x[n - 1] = \frac{1}{2}x[n - 1] + \frac{1}{2}(-1)^n x[n - 1].$$

$$x[n - 1] \longleftrightarrow a_k e^{-jk\omega_0(1)} \quad (\text{Time Shifting Property}) \quad (1)$$

Note that $(-1)^n = (e^{j\pi})^n = e^{j4(\frac{2\pi}{8})n} = e^{j4\omega_0 n}$, $\omega_0 = 2\pi/8 = \frac{\pi}{4}$

$$(-1)^n x[n - 1] = e^{j4\omega_0 n} x[n - 1] \longleftrightarrow a_{k-4} e^{-j(k-4)\omega_0} \quad (\text{Frequency Shifting Property}) \quad (2)$$

From (1) and (2): $y[n] = \frac{1}{2}x[n - 1] + \frac{1}{2}(-1)^n x[n - 1] \longleftrightarrow b_k = \frac{1}{2}a_k e^{-jk\omega_0} + \frac{1}{2}a_{k-4} e^{-j(k-4)\omega_0}$.
Substituting $a_k = -a_{k-4}$:

$$\begin{aligned} b_k &= \frac{1}{2}a_k e^{-jk\omega_0} + \frac{1}{2}(-a_k) e^{-jk\omega_0} e^{j4\omega_0} \\ &= \frac{1}{2}a_k e^{-jk\omega_0} (1 - e^{j4\omega_0}) = \frac{(1 - e^{j4\omega_0})}{2} e^{-jk\omega_0} a_k \\ &= \frac{(1 - e^{j4\frac{\pi}{4}})}{2} e^{-jk\frac{\pi}{4}} a_k = \frac{1 - (-1)}{2} e^{-jk\frac{\pi}{4}} a_k \\ &= e^{-jk\frac{\pi}{4}} a_k \end{aligned}$$

$$\rightarrow f[k] = e^{-jk\frac{\pi}{4}} .$$