Signals and Systems

Fall 2003

Lecture #5

18 September 2003

1. Complex Exponentials as Eigenfunctions of LTI Systems
2. Fourier Series representation of CT periodic signals
3. How do we calculate the Fourier coefficients?
4. Convergence and Gibbs' Phenomenon

Portrait of Jean Baptiste Joseph Fourier

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Signals & Systems, 2nd ed. Upper Saddle River, N.J.: Prentice Hall, 1997, p. 179.

Desirable Characteristics of a Set of “Basic” Signals

a. We can represent large and useful classes of signals using these building blocks

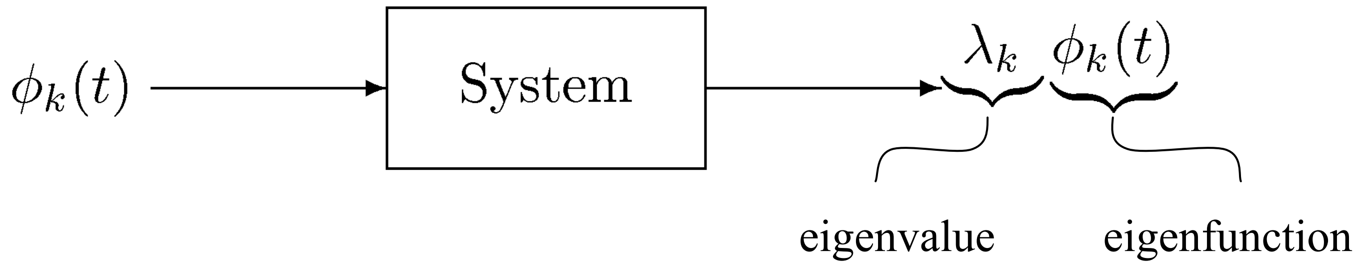
b. The response of LTI systems to these basic signals is particularly simple, useful, and insightful

Previous focus: Unit samples and impulses

Focus now: Eigenfunctions of all LTI systems

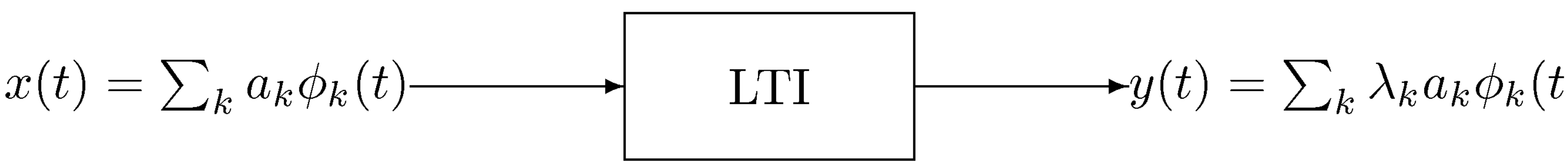
The eigenfunctions $\phi_k(t)$ and their properties

(Focus on CT systems now, but results apply to DT systems as well.)



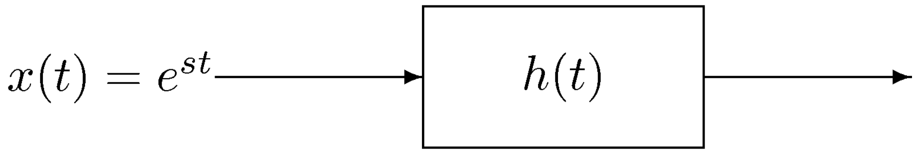
Eigenfunction in → same function out with a "gain"

From the superposition property of LTI systems:

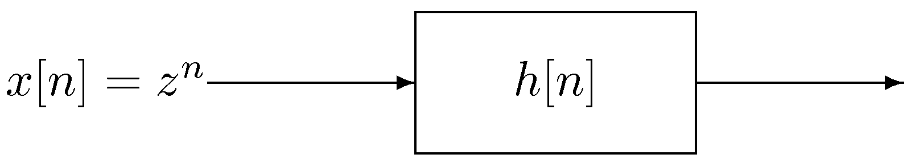


Now the task of finding response of LTI systems is to determine λ_k.

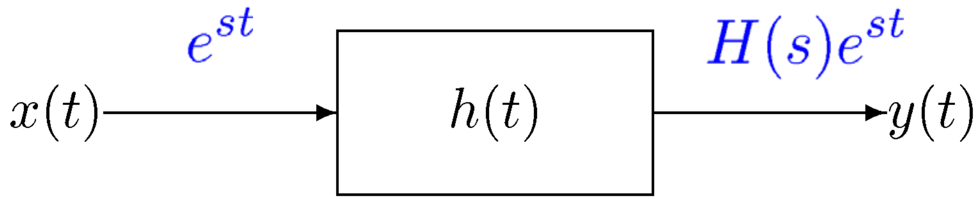
Complex Exponentials as the Eigenfunctions of any LTI Systems



$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= \left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right] e^{st} \\
&= \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}}
\end{aligned}
$$



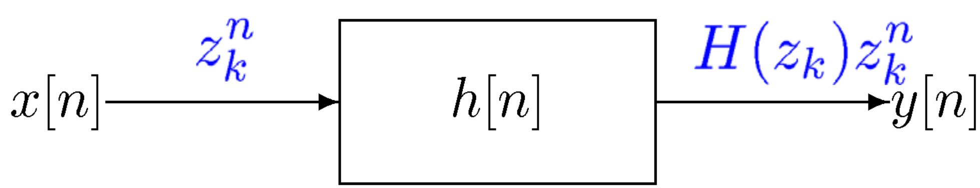
$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} h[m] z^{n-m} \\
&= \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right] z^n \\
&= \underbrace{H(z)}_{\text{eigenvalue}} \underbrace{z^n}_{\text{eigenfunction}}
\end{aligned}
$$



$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

$$x(t) = \sum_k a_k e^{s_k t} \longrightarrow y(t) = \sum_k H(s_k) a_k e^{s_k t}$$

DT:



$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$x[n] = \sum_k a_k z_k^n \longrightarrow y[n] = \sum_k H(z_k) a_k z_k^n$$

What kinds of signals can we represent as "sums" of complex exponentials?

For Now: Focus on restricted sets of complex exponentials

CT: $s = jw$ – purely imaginary, i.e., signals of the form $e^{j\omega t}$

DT: $z = e^{jw}$, i.e., signals of the form $e^{j\omega n}$

Magnitude 1

$\Downarrow$

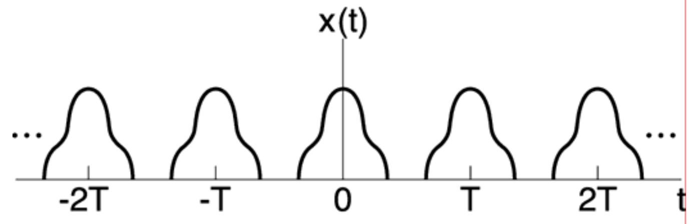
CT & DT Fourier Series and Transforms

Periodic Signals

Fourier Series Representation of CT Periodic Signals

$x(t) = x(t+T)$ for all t

- smallest such T is the *fundamental period*
- $\omega_o = \dfrac{2\pi}{T}$ is the *fundamental frequency*



$$e^{j\omega t} \text{ periodic with period } T \Leftrightarrow \omega = k\omega_0$$

$$\Downarrow$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}$$

- periodic with period T
- $\{a_k\}$ are the *Fourier (series) coefficients*
- $k = 0$ DC
- $k = \pm 1$ first harmonic
- $k = \pm 2$ second harmonic

Question #1: How do we find the Fourier coefficients?

First, for simple periodic signals consisting of a few sinusoidal terms

Ex: $x(t) = \cos 4\pi t + 2\sin 8\pi t$

Euler's relation (memorize!)

$$= \frac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right] + \frac{2}{2j}\left[e^{j8\pi t} - e^{-j8\pi t}\right]$$

$$\omega_0 = 4\pi \qquad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$a_0 = 0$ – no dc component

$$a_1 = \frac{1}{2}$$

$$a_{-1} = \frac{1}{2}$$

$$a_2 = \frac{1}{j}$$

$$a_{-2} = -\frac{1}{j}$$

$$a_3 = 0$$

$$a_{-3} = 0$$

$$\vdots$$

- For *real* periodic signals, there are two other commonly used forms for CT Fourier series:

$$x(t) = a_0 + \sum_{k=1}^{\infty} [\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t]$$

or

$$x(t) = a_0 + \sum_{k=1}^{\infty} [\gamma_k \cos(k\omega_0 t + \theta_k)]$$

- Because of the eigenfunction property of $e^{j\omega t}$, we will usually use the complex exponential form in 6.003.

  - A consequence of this is that we need to include terms for *both* positive and negative frequencies:

$$e^{jk\omega_0 t} \quad , \quad e^{-jk\omega_0 t}$$

$$\text{Remember } \cos(k\omega_0 t) = \frac{1}{2}(e^{jk\omega_0 t} + e^{-jk\omega_0 t})$$

$$\text{and } \sin(k\omega_0 t) = \frac{1}{2j}(e^{jk\omega_0 t} - e^{-jk\omega_0 t})$$

Now, the complete answer to Question #1

(Given $x(t)$, how find a_k?)

Suppose

1) multiply by $e^{-jn\omega_0 t}$
2) integrate over one period

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

1) multiply by $e^{-jn\omega_0 t}$
2) integrate over one period

$$\Downarrow$$

$$\int_T x(t)e^{-jn\omega_0 t}dt = \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right) e^{-jn\omega_0 t}dt$$

$$= \sum a_k \left(\int_T e^{j(k-n)\omega_0 t}dt\right)$$

(Here $\int_T$ denotes integral over *any* interval of length T (one period).)

Next, note that

$$\int_T e^{j(k-n)\omega_0 t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$= T\delta[k-n] \quad \text{Orthogonality}$$

$$\Downarrow$$

$$\int_T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \left(\int_T e^{j(k-n)\omega_0 t}dt\right) = \sum_{k=-\infty}^{\infty} a_k \cdot T\delta[k-n]$$

$$\int_T x(t)e^{-jn\omega_0 t}dt = a_n T$$

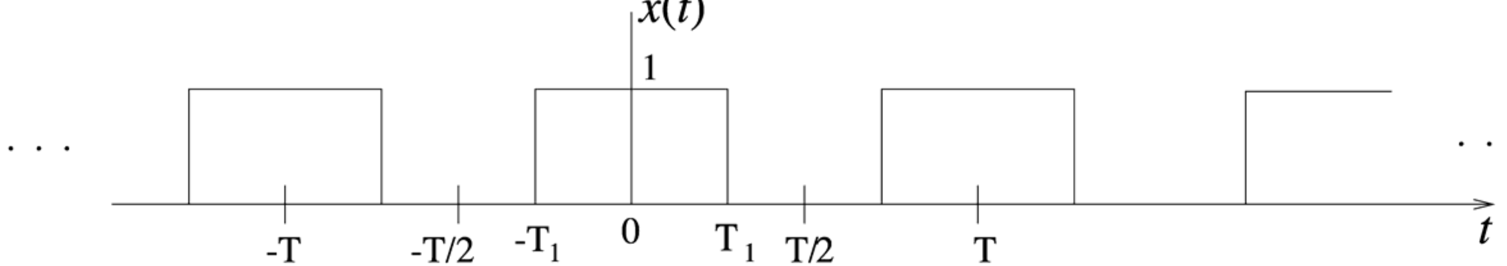
$$\Downarrow$$

CT Fourier Series Pair $\left(\omega_0 = \frac{2\pi}{T}\right)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad \text{(Analysis equation)}$$

Ex: Periodic Square Wave



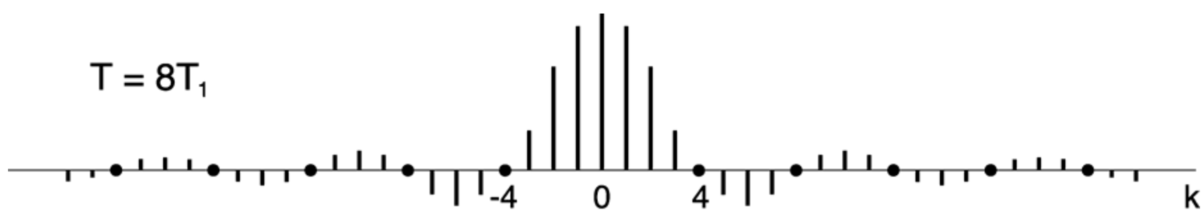
For $k = 0$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{2T_1}{T}$$

DC component is just the average

For $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt \\ &= -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{\sin k\omega_0 T_1}{k\pi} \qquad \left(\omega_0 = \frac{2\pi}{T}\right) \end{aligned}$$



Convergence of CT Fourier Series

- How can the Fourier series for the square wave possibly make sense?
- The key is: What do we *mean* by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad ?$$

- One useful notion for engineers: there is no *energy* in the difference

$$e(t) = x(t) - \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\int_T |e(t)|^2 dt = 0$$

(just need $x(t)$ to have finite energy per period)

$$\int_T |x(t)|^2 dt < \infty$$

Under a different, but reasonable set of conditions (the Dirichlet conditions)

Condition 1. $x(t)$ is *absolutely integrable* over one period, i. e.

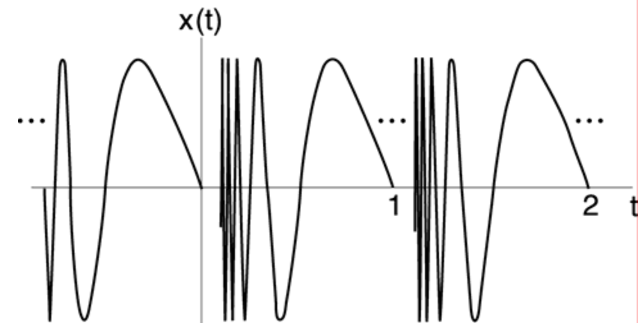
$$\int_T |x(t)|dt < \infty$$

And

Condition 2. In a finite time interval, $x(t)$ has a *finite* number of maxima and minima.

Ex. An example that violates Condition 2.

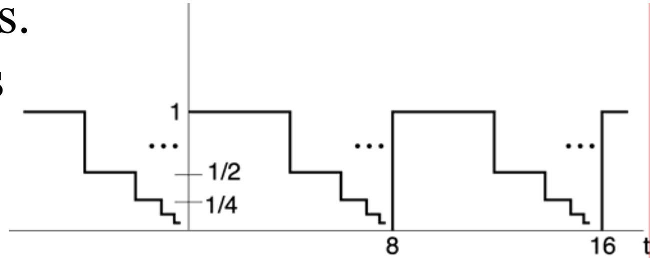
$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \le 1$$



And

Condition 3. In a finite time interval, $x(t)$ has only a *finite* number of discontinuities.

Ex. An example that violates Condition 3.



- Dirichlet conditions are met for the signals we will encounter in the real world. Then

  - The Fourier series = $x(t)$ at points where $x(t)$ is continuous

  - The Fourier series = "midpoint" at points of discontinuity

- Still, convergence has some interesting characteristics:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

  - As $N \to \infty$, $x_N(t)$ exhibits *Gibbs*' phenomenon at points of discontinuity

Demo: Fourier Series for CT square wave (Gibbs phenomenon).