MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

0.4pt0pt

6.003: Signals and Systems—Fall 2003

Quiz 1
Tuesday, October 14, 2003

Directions: The exam consists of 5 problems on pages 2 to 19 and work space on pages 20 and 21. Please make sure you have all the pages. Tables of Fourier series properties are supplied to you at the end of this booklet. **Enter all your work and your answers directly in the spaces provided on the printed pages of this booklet. Please make sure your name is on all sheets. DO IT NOW!**. All sketches must be adequately labeled. Unless indicated otherwise, **answers must be derived or explained**, not just simply written down. This examination is closed book, but students may use one 8 1/2 $\times$ 11 sheet of paper for reference. Calculators may not be used.

NAME: SOLUTIONS______________________________

Check your section	Section	Time	Rec. Instr.
☐	1	10-11	Prof. Zue
☐	2	11-12	Prof. Zue
☐	3	1- 2	Prof. Gray
☐	4	11-12	Dr. Rohrs
☐	5	12- 1	Prof. Voldman
☐	6	12- 1	Prof. Gray
☐	7	10-11	Dr. Rohrs
☐	8	11-12	Prof. Voldman

Please leave the rest of this page blank for use by the graders:

Problem	**No. of points**	**Score**	**Grader**
1	18		
2	20		
3	20		
4	21		
5	21		
Total	100		

PROBLEM 1 (18%)

For the questions in this problem, no explanation is necessary.

Consider the following three systems:

SYSTEM A: $y(t) = x(t+2)\sin(\omega t + 2),$ where $\omega \neq 0$

SYSTEM B: $y[n] = \left(-\frac{1}{2}\right)^n (x[n] + 1)$

SYSTEM C: $y[n] = \sum_{k=1}^{n} \left(x^2[k+1] - x[k]\right)$

where x and y are the input and output of each system.

Circle YES or NO for each of the following questions for each of these three systems.

	SYSTEM A	SYSTEM B	SYSTEM C
Is the system linear ?	[YES] NO	YES [NO]	YES [NO]
Is the system time invariant ?	YES [NO]	YES [NO]	YES [NO]
Is the system causal ?	YES [NO]	[YES] NO	YES [NO]
Is the system stable ?	[YES] NO	YES [NO]	YES [NO]

Work Page for Problem 1
Linearity

System A: YES

$$\begin{aligned} x_1(t) \rightarrow y_1(t) &= x_1(t+2)sin(wt+2) \\ x_2(t) \rightarrow y_2(t) &= x_2(t+2)sin(wt+2) \\ x_3(t) = ax_1(t) + bx_2(t) \rightarrow y_3(t) &= x_3(t+2)sin(wt+2) \\ &= ax_1(t+2)sin(wt+2) + bx_2(t+2)sin(wt+2) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

System B: NO if $x[n] = 0$, then $y[n] \neq 0$

System C: NO not linear due to the $x^2[k+1]$ term.

Time Invarience

System A: NO the output has time varying gain

System B: NO the output has time varying gain

System C: NO the number of summed terms depends on n

Causality

System A: NO $y(t)$ depends on $x(t+2)$

System B: YES $y[n]$ depends only on the current value of $x[n]$

System C: NO $y[n]$ depends on the future value of $x[n]$

Stability

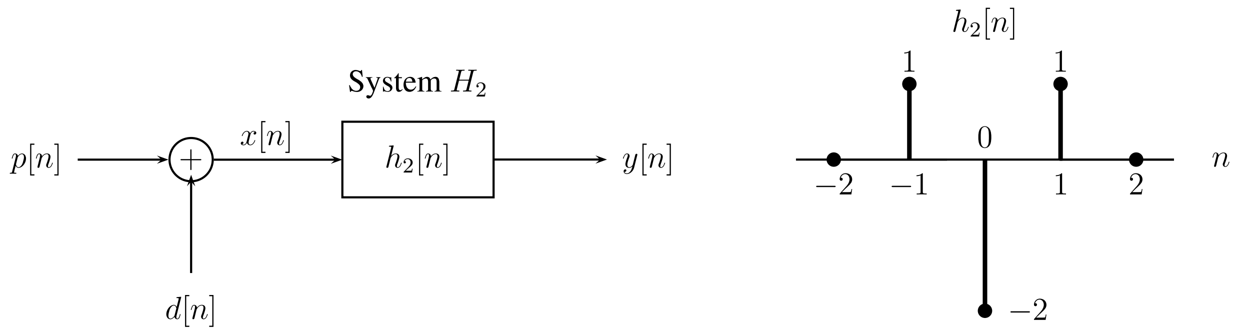
System A: YES bounded $x(t) \longrightarrow$ bounded $y(t)$

System B: NO $(-\frac{1}{2})^n \longrightarrow \infty$ as $n \longrightarrow -\infty$

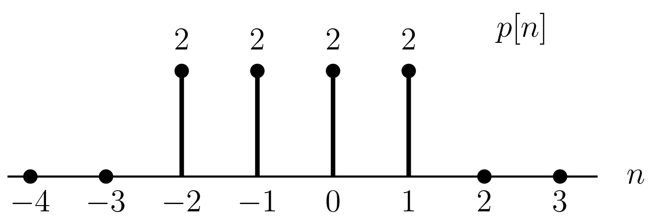
System C: NO if $x[n] = -1 \longrightarrow y[n] = \sum_{k=1}^{n} 2$, $y \longrightarrow \infty$ as $n \longrightarrow \infty$

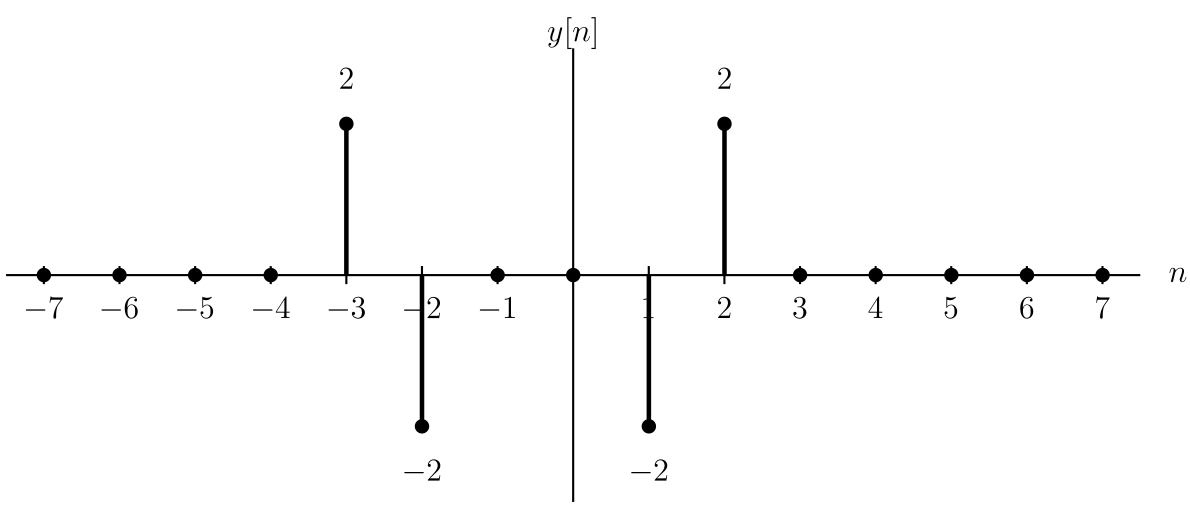
PROBLEM 2 (20%)

Consider a DT LTI system, H_2 with a unit sample response $h_2[n] = h[n] * h[n+1]$, as shown below, where $h[n] = \delta[n] - \delta[n-1]$. You may remember from one of the lectures that $h[n]$ can be viewed as the unit sample response of a DT LTI system that acts as an edge detector. The purpose of this problem is to develop an edge detector that is robust against additive noise.



Part a. Assume that the input to the system, $p[n]$ is as shown below, and there is no noise, i.e., $d[n] = 0$ and $p[n] = x[n]$. Provide a labeled sketch of $y[n]$, the output of the system.

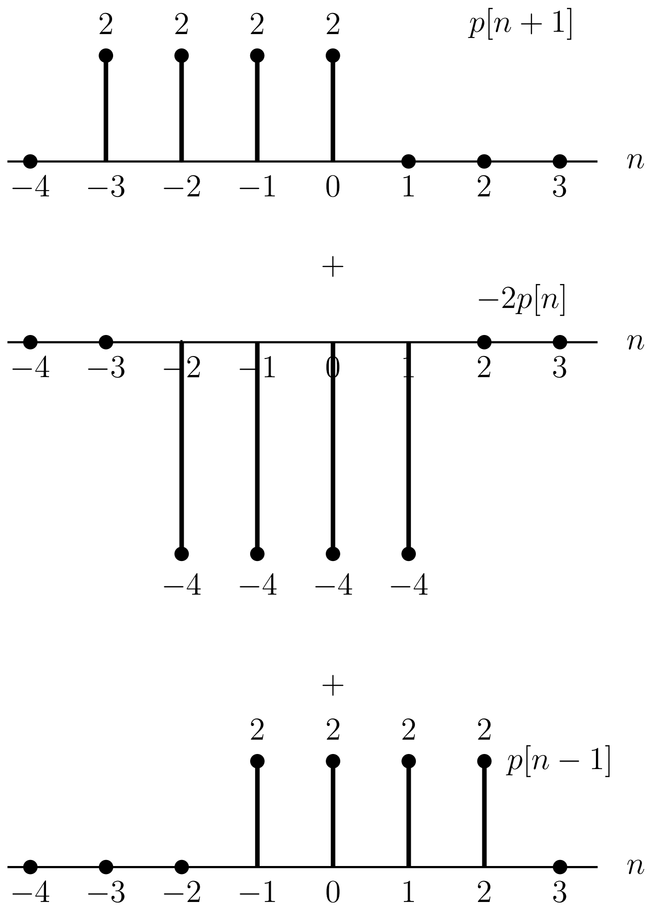




Work Page for Problem 2

$x[n] = p[n], \qquad y[n] = x[n] * h_2[n]$

Using DT convolution sum, $y[n]$ is the sum of 3 copies of $p[n]$ shifted and scaled according to $h_2[n]$

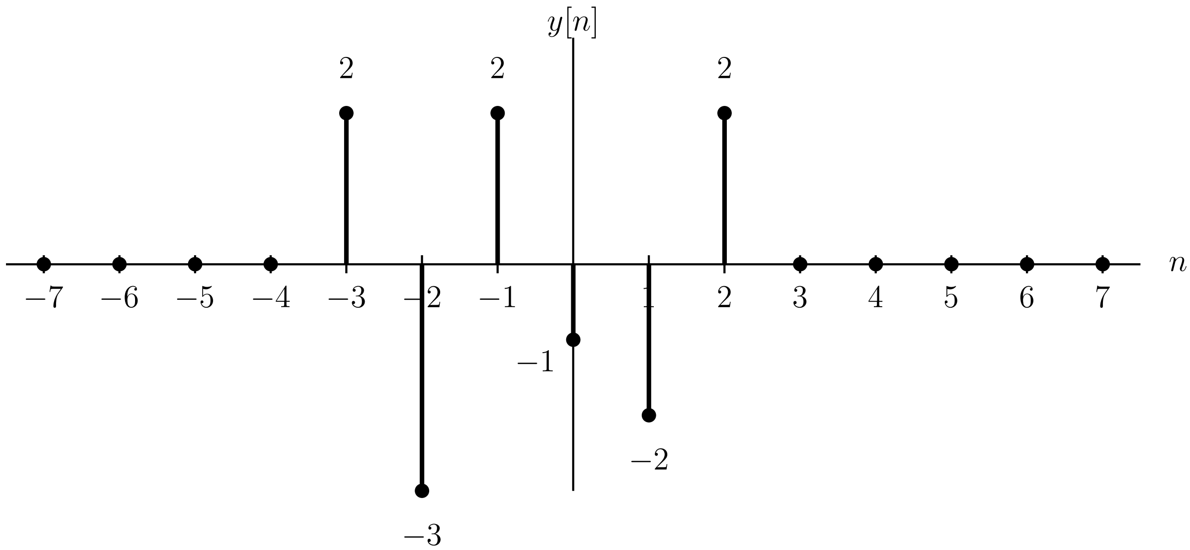


The resulting $y[n]$ is shown on page 4.

Part b. For the same input signal as **Part a.**, now assume that the noise signal is

$$d[n] = -\delta[n+1].$$

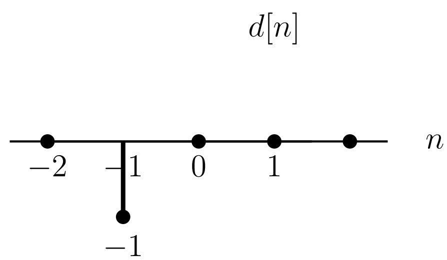
Provide a labeled sketch of the output $y[n]$, i.e., the response to $x[n] = p[n] + d[n]$.

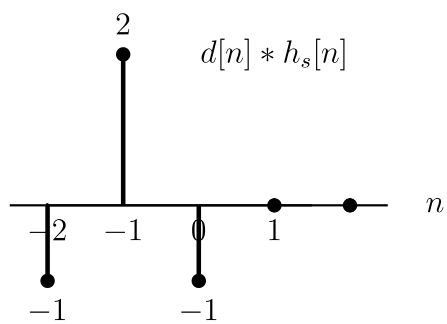


Work Page for Problem 2

$$
\begin{aligned}
y[n] &= x[n] * h_2[n] \\
&= (p[n] + d[n]) * h_2[n] \\
&= (p[n] * h_2[n]) + (d[n] * h_2[n])
\end{aligned}
$$

We have already found $(p[n] * h_2[n])$ in **Part a**. Now we need $d[n] * h_2[n]$.

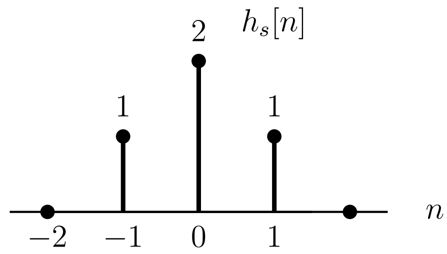


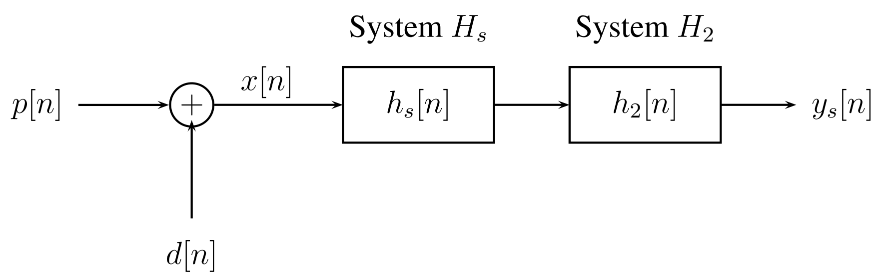


Adding the above $(d[n] * h_2[n])$ to the answer in **Part a**, we get $y[n]$ for this part which is shown on page 6.

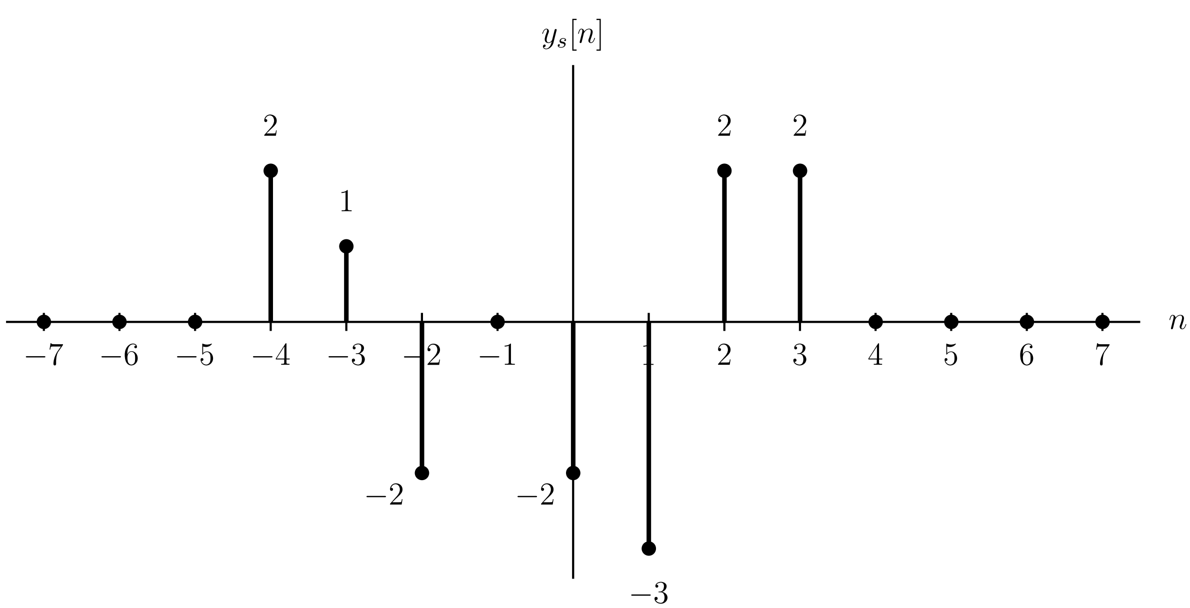
 Problem 2 continues on the following page.

Part c. In order to use system H_2 as a part of an edge detector, we would like to add an LTI system H_s whose unit sample response, $h_s[n]$ is shown below. System H_s smoothes out effect of noise on $x[n]$. The overall system can be represented as below:





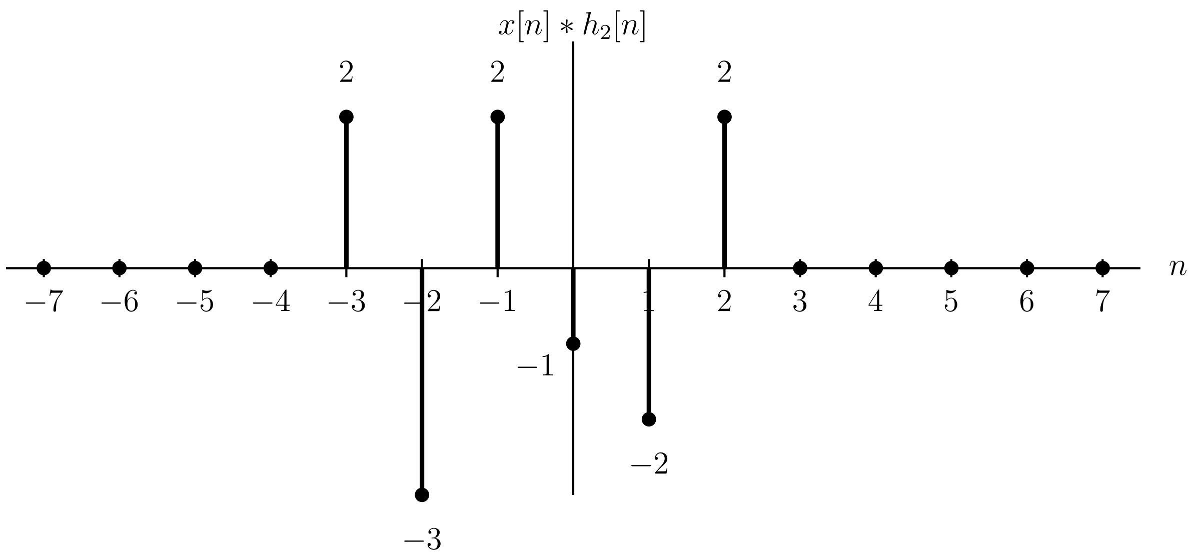
Provide a labeled sketch of the overall output $y_s[n]$, when $p[n]$ and $d[n]$ are exactly the same as in **Part b.**



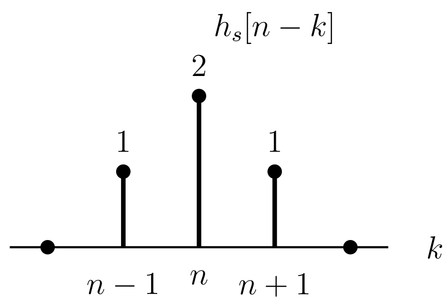
Work Page for Problem 2

$$
\begin{aligned}
y_s[n] &= x[n] * h_s[n] * h_2[n] \\
&= (x[n] * h_2[n]) * h_s[n]
\end{aligned}
$$

$x[n] = p[n] + d[n]$ is as defined in **Part b**. Therefore, we already know $(x[n] * h_2[n])$ from **Part b**.



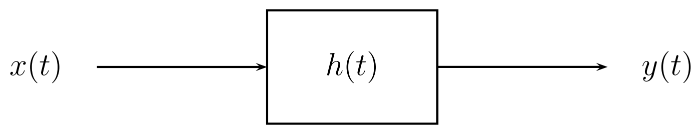
Now we just need to convolve the above signal with $h_s[n]$. We can do this convolution with flipping and sliding $h_s[n]$.

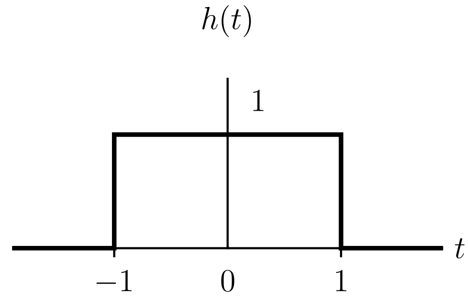


The result is the desired $y_s[n]$ and is shown on page 8.

PROBLEM 3 (20%)

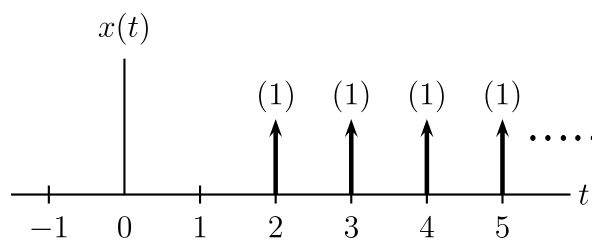
Consider the CT LTI system whose impulse response is given as:



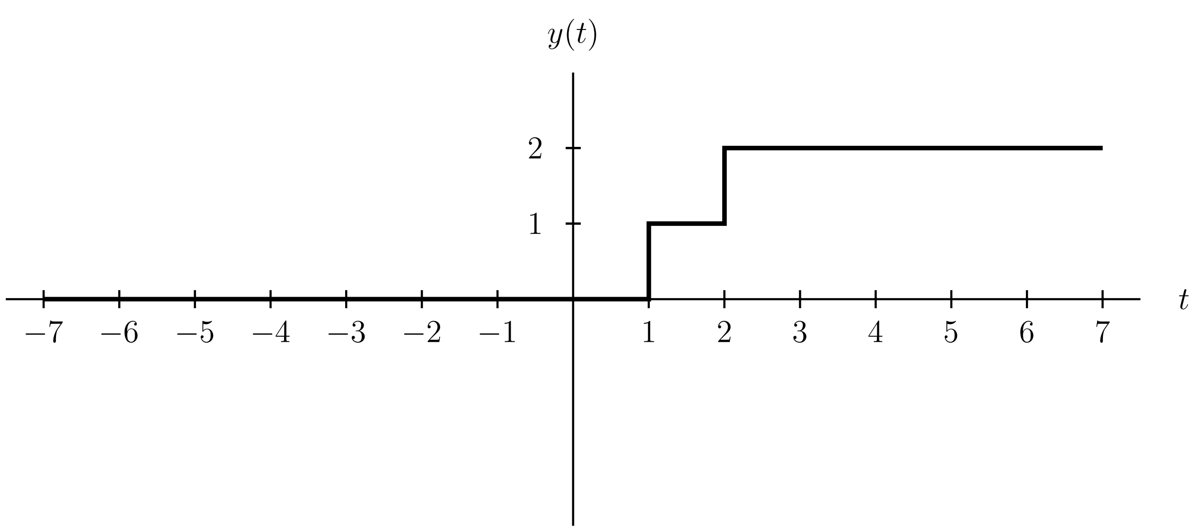


The following two parts can be done independently.

Part a. The input $x(t)$, an impulse train starting at $t = 2$, is depicted below:

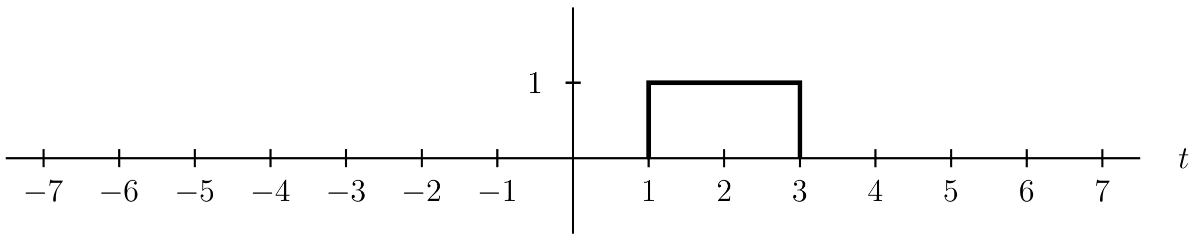


Provide a labeled sketch of the corresponding output $y(t)$.



Work Space for Problem 3

Using CT convolution sum (summation of shifted and scaled impulse response), we can find $y(t)$.



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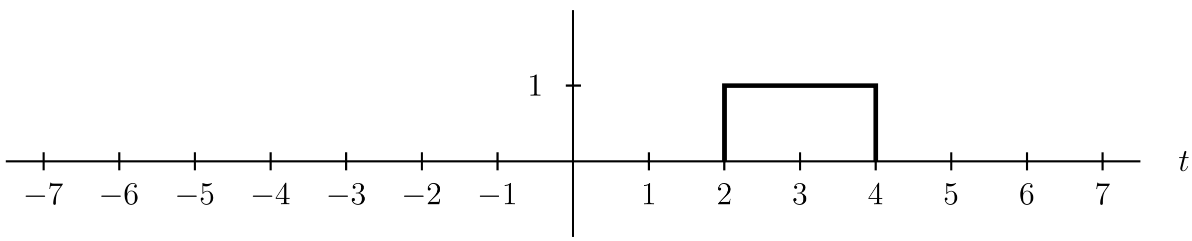

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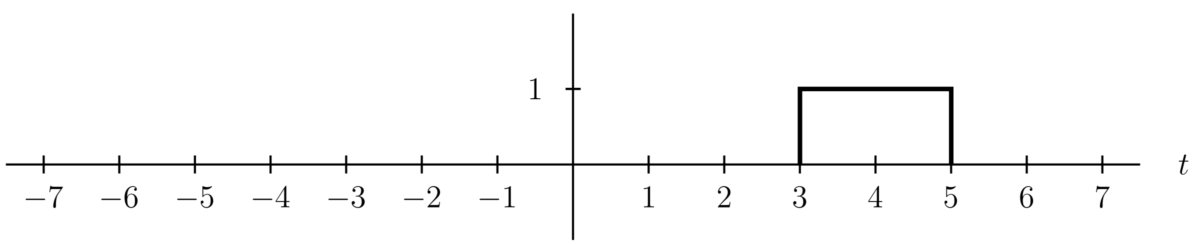

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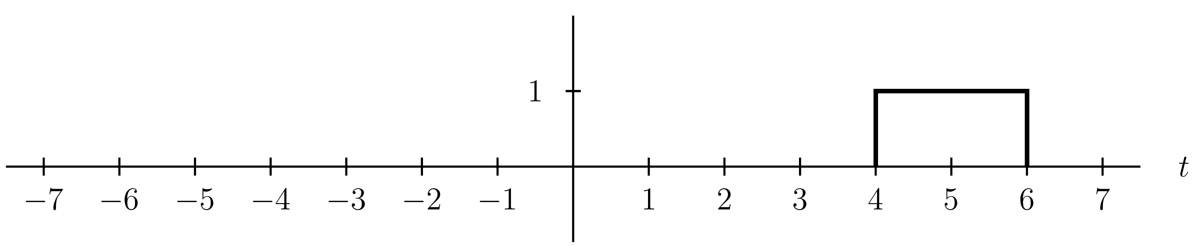

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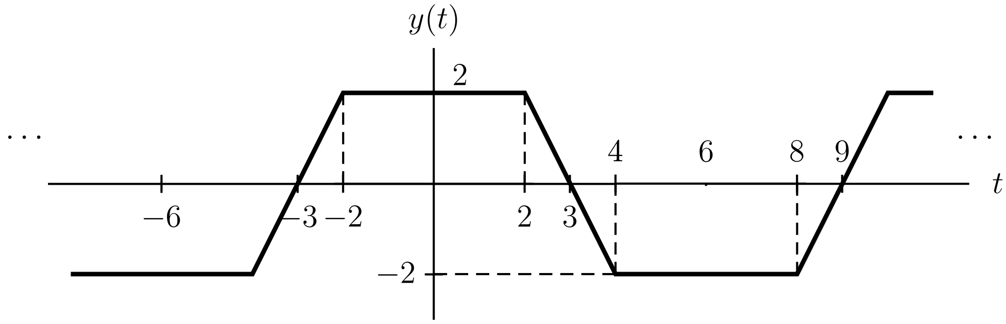
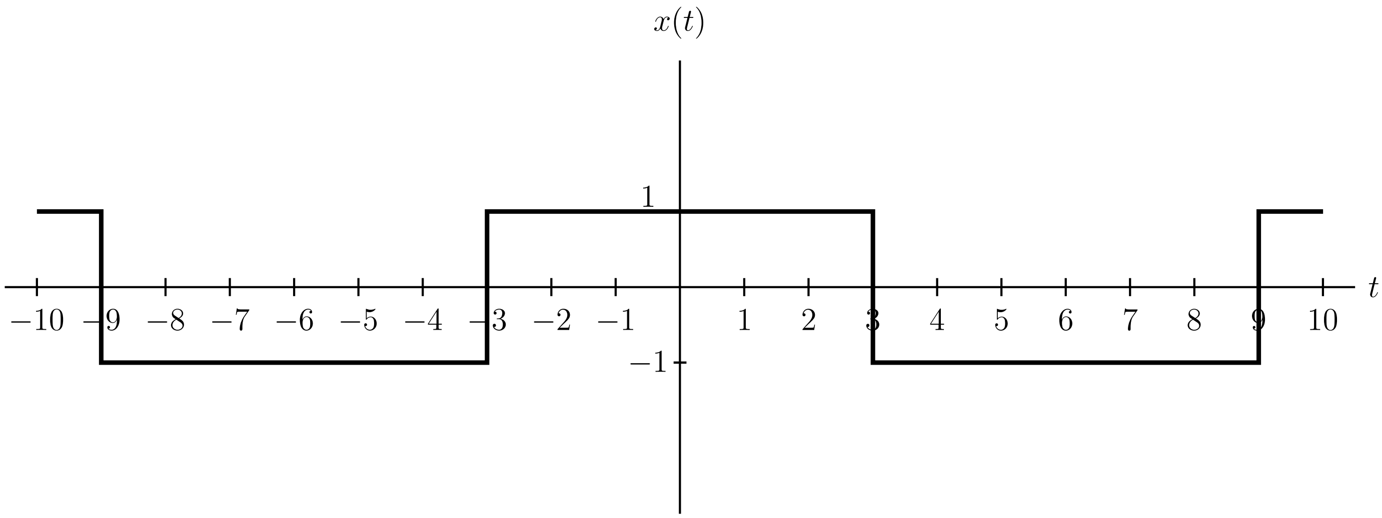
Adding the shifted and scaled pulses, we get $y(t)$ as shown on page 10.

 Problem 3 continues on the following page.

Part b. For this part, the output $y(t)$ is periodic and is depicted below:

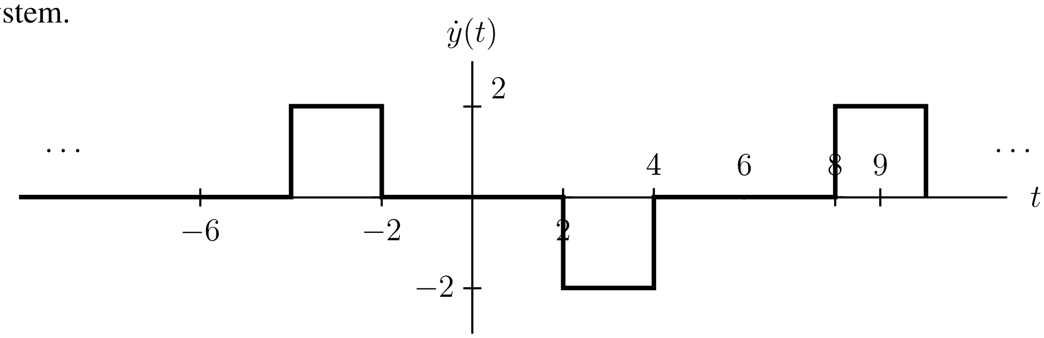


Provide a labeled sketch of the input $x(t)$ that produces this $y(t)$.

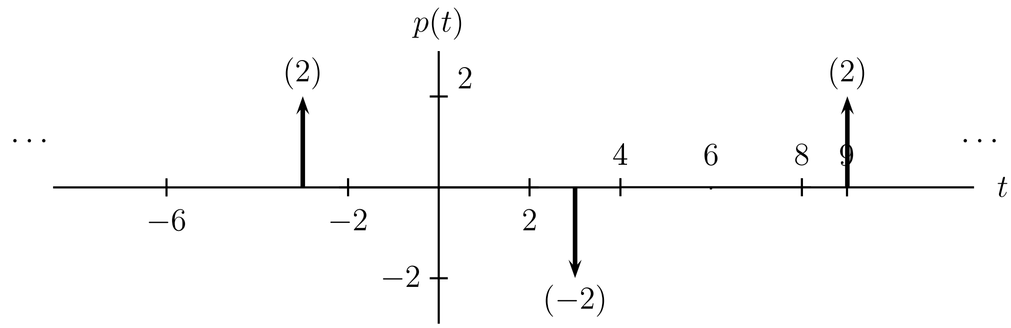


Work Page for Problem 3

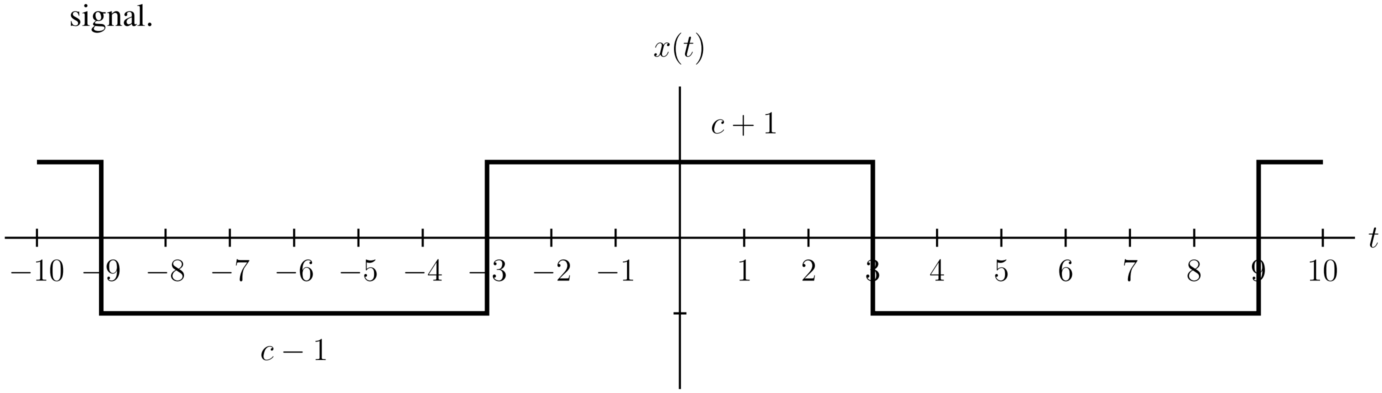
There are different ways to solve this problem, and here is one of them:
Let's take the derivative of the output, $\frac{dy(t)}{dt}$, and relate it to the impulse response of the system.



It can be seen that $\dot{y}(t)$ is the sum of a number of scaled and shifted versions of h(t). Alternatively,: $\dot{y}(t) = p(t) * h(t)$, where $p(t)$ is shown below:



We know that $y(t) = x(t) * h(t) \Rightarrow x(t) = u_{-1}(t) * p(t)$
$\Rightarrow x(t) = \int_{-\infty}^{t} p(v)dv = \hat{x}(t) + c$, where the constant c restores the DC level of the periodic signal.

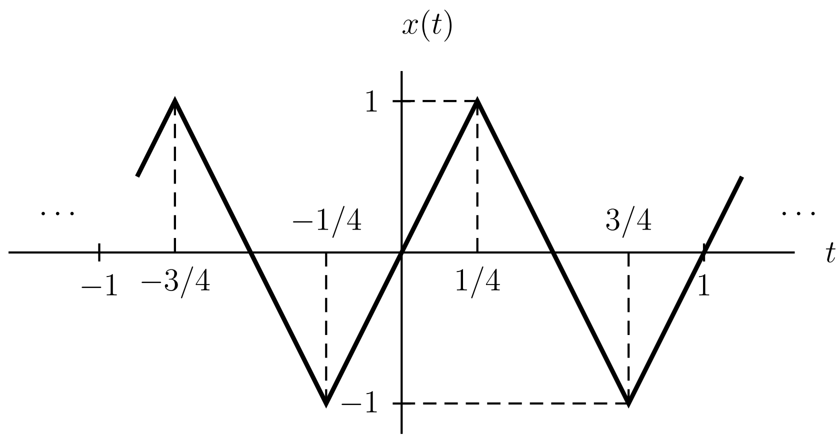


To find the value of the constant c, we compare the value of $y(0)$ with $x(t) * h(t)|_{t=0}$, i.e. the area under the curve of x(t)h(-t).

$$y(0) = 2 = x(t) * h(t)|_{t=0} = (1 + c)(2) = 2 \rightarrow c = 0.$$

PROBLEM 4 (21%)

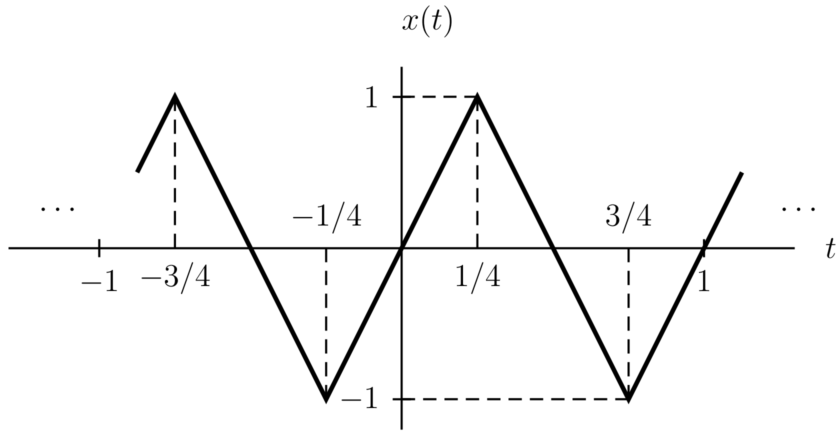
Consider the following periodic triangular wave shown below:



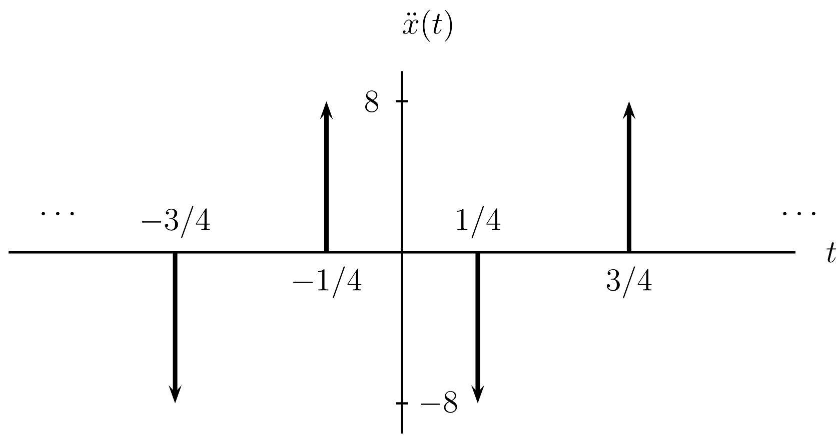
Part a. Determine the Fourier series coefficients, a_k for $x(t)$.

$$a_k = \begin{cases} 0, & k = 0 \\ \frac{-4j}{k^2\pi^2}\sin(k\frac{\pi}{2}), & k \neq 0 \end{cases}$$

Work Page for Problem 4



Period= $T = 1 \rightarrow \omega_0 = 2\pi/T = 2\pi$.
$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{T}$(Area under the curve for one period)= $\boxed{0 = a_0}$
Finding $a_{k\neq 0}$ using the analysis equation will be tedious. Instead, we will use the integration property of the Fourier series.



Let $\ddot{x}(t) \leftrightarrow b_k$

$$\begin{aligned} b_k &= \frac{1}{T}\int_T \ddot{x}(t)e^{-jk\omega_0 t}dt = \int_{-1/2}^{1/2} \left[8\delta(t+1/4) - 8\delta(t-1/4)\right] e^{-jk\omega_0 t}dt \\ &= 8e^{-jk\omega_0(-1/4)} - 8e^{-jk\omega_0(1/4)} = 16j\sin(k\omega_0/4) \\ &= 16j\sin(k\frac{\pi}{2}). \\ a_k &= \frac{1}{(jk\omega_0)^2}b_k \quad \text{(integration property)} \\ &= \frac{-16j}{k^2(2\pi)^2}\sin(k\frac{\pi}{2}) = \boxed{\frac{-4j}{k^2\pi^2}\sin(k\frac{\pi}{2}).} \end{aligned}$$

As a double check, note that $a_{-k} = a_k^*$ because $x(t)$ is real.

 Problem 4 continues on the following page.

Part b. Consider a causal LTI system, S, whose input-output relation is characterized by the following stable linear constant coefficient differential equation:

$$\frac{d^2y}{dt^2} + 4\pi\frac{dy}{dt} + 4\pi^2 y(t) = 4\pi^2 x(t),$$

where $x(t)$ is the input and $y(t)$ is the output of the system. Suppose $x(t)$ shown on the previous page is applied to the system S as an input. Let b_k be the Fourier coefficients of the corresponding output $y(t)$. Find b_3 and b_{-3}.

$$b_3 = \frac{4j}{9\pi^2(-8+j6)} \qquad b_{-3} = \frac{4j}{9\pi^2(8+j6)}.$$

Work Page for Problem 4

First, we find $H(j\omega)$ by inspection from the coefficients of the differential equation:

$$\frac{d^2y}{dt^2} + 4\pi\frac{dy}{dt} + 4\pi^2 y(t) = 4\pi^2 x(t),$$

$$H(j\omega) = \frac{4\pi^2}{(j\omega)^2 + 4\pi j\omega + 4\pi^2} = \frac{1}{1 + j\omega/\pi - \omega^2/4\pi^2}.$$

$\because b_k = a_k H(jk\omega_0)$, where $\omega_0 = 2\pi$, therefore:

$$\begin{aligned}
b_3 &= a_k H(jk\omega_0)\,|_{k=3} = a_3 H(j6\pi) \\
&= \left(\frac{-4j}{(3)^2\pi^2}\sin\left(3\frac{\pi}{2}\right)\right)\frac{1}{1 + j6\pi/\pi - (6\pi)^2/4\pi^2} = \frac{4j}{9\pi^2}\frac{1}{(1 + j6 - 9)} \\
b_3 &= \boxed{\frac{4j}{9\pi^2(-8+6j)}.} \\
b_{-3} &= b_3^* = \frac{-4j}{9\pi^2}\frac{1}{-8-6j} = \boxed{\frac{4j}{9\pi^2(8+6j)}.}
\end{aligned}$$

PROBLEM 5 (21%)

You are given the following facts about a discrete time sequence $x[n]$:

(a) $x[n]$ is real and odd.

(b) $x[n]$ is periodic with period $N = 6$.

(c) $\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = 10.$

(d) $\sum_{n=<N>} (-1)^{n/3} x[n] = 6j.$

(e) $x[1] > 0.$

Find an expression of $x[n]$ in the form of sines and cosines.

There are two possible answers
(depending on whether condition (d) was used to find a_1 or a_{-1}):

$$x[n] = 4\sin(\tfrac{2\pi}{3}n) + 2\sin(\tfrac{\pi}{3}n)$$

or

$$x[n] = 4\sin(\tfrac{2\pi}{3}n) - 2\sin(\tfrac{\pi}{3}n)$$

Work Page for Problem 5

To Find an expression of $x[n]$ in the form of sines and cosines, we first need to find a_k, the Fourier series coefficients of $x[n]$.

Here we will show the information we can extract from each given fact:

(a) $x[n]$ is real and odd. $\Rightarrow a_k$ is pure imaginary and odd, and $\boxed{a_{-k} = a_k^*}$

$\because a_k$ is odd $\rightarrow \boxed{a_0 = 0}$

(b) $x[n]$ is periodic with period $N = 6 \rightarrow \omega_0 = 2\pi/6 = \pi/3$.

$\because N = 6 \rightarrow a_k = a_{k+N} \rightarrow a_{-3} = a_3 \qquad (5.1)$

$\because a_k$ is odd $\rightarrow a_{-k} = -a_k \rightarrow a_{-3} = -a_3 \qquad (5.2)$

From (5.1) and (5.2) we can conclude that $\boxed{a_{-3} = a_3 = 0}$.

Another result that we can conclude in this part is that a_k has only 6 distinct values. So the only values we still need to find are $a_{\pm 2}$ and $a_{\pm 1}$

(c) $\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = 10.$

By Perseval's therom: $\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$

$$\begin{aligned} \Rightarrow |a_{-2}|^2 + |a_{-1}|^2 + |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 &= 10 \\ |a_{-2}|^2 + |a_{-1}|^2 + |a_1|^2 + |a_2|^2 &= 10 \\ \because a_{-k} = a_k^* \Rightarrow 2|a_2|^2 + 2|a_1|^2 &= 10 \qquad (*) \end{aligned}$$

(d) $\sum_{n=<N>} (-1)^{n/3} x[n] = 6j.$

Remembering that $e^{\pm j\pi} = -1$

$$\begin{aligned} \Rightarrow \sum_{n=<N>} (-1)^{n/3} x[n] &= \sum_{n=<N>} (e^{\pm j\pi})^{n/3} x[n] \\ &= \sum_{n=<N>} e^{\pm j\pi n/3} x[n] \\ &= N\left(\frac{1}{N}\sum_{n=<N>} x[n]\, e^{-jk\omega_0 n}\right)_{k=\mp 1} \\ &= N a_{\mp 1} = 6 a_{\mp 1} = 6j \end{aligned}$$

$\Rightarrow a_{\mp 1} = j$ and $a_{\pm 1} = -j \rightarrow \boxed{a_1 = \pm j \text{ and } a_{-1} = \mp j}$

From (*): $2|a_2|^2 + 2|a_1|^2 = 2|a_2|^2 + 2(1)^2 = 10 \Rightarrow \boxed{|a_2| = 2}$

(e) $x[1] > 0$.

$\because a_k$ is odd $\rightarrow a_{-k} = -a_k$.

$\because |a_2| = 2 \rightarrow$ either $a_2 = 2j \rightarrow a_{-2} = -2j$ or $a_2 = -2j \rightarrow a_{-2} = 2j$

To find which choice is the right one, let's designate p to be the sign of a_2.

$\rightarrow a_2 = 2jp$, where $p = 1$ if $a_2 = 2j$ and $p = -1$ if $a_2 = -2j$.

$$\begin{aligned} x[n] &= \sum_{k=<N>} a_k e^{jk\omega_0 n} \\ &= a_{-2}e^{j(-2)\omega_0 n} + a_{-1}e^{j(-1)\omega_0 n} + a_1 e^{j(1)\omega_0 n} + a_2 e^{j(2)\omega_0 n} \\ &= -a_2 e^{-j2\omega_0 n} - a_1 e^{-j\omega_0 n} + a_1 e^{j\omega_0 n} + a_2 e^{j2\omega_0 n} \\ &= a_2 e^{j2\omega_0 n} - a_2 e^{-j2\omega_0 n} + a_1 e^{j\omega_0 n} - a_1 e^{-j\omega_0 n} \\ &= a_2\left(e^{j2\omega_0 n} - e^{-j2\omega_0 n)}\right) + a_1\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right) \\ &= a_2(2j)\sin(2\omega_0 n) + a_1(2j)\sin(\omega_0 n) \\ &= (2jp)(2j)\sin(2\omega_0 n) + (\pm j)(2j)\sin(\omega_0 n) \\ &= -4p\sin(2\omega_0 n) \mp 2\sin(\omega_0 n) \end{aligned}$$

$$\begin{aligned} x[1] &= -4p\sin(2\omega_0) \mp 2\sin(\omega_0) = -4p\sin(2\pi/3) \mp 2\sin(\pi/3) \\ &= (-4p \mp 2)\frac{\sqrt{3}}{2} \end{aligned}$$

$\because x[1] > 0 \rightarrow -4p \mp 2 > 0 \rightarrow p = -1$, regardless of the sign used for the second term.

$\Rightarrow \boxed{x[n] = 4\sin\left(\frac{2\pi}{3}n\right) \mp 2\sin\left(\frac{\pi}{3}n\right)}$

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