

6.003: Signals and Systems—Fall 2003

PROBLEM SET 5 SOLUTIONS

Issued: October 22, 2003

Exercise for home study:

O&W 4.47

This problem examines the Fourier transform of a continuous-time LTI system with a real, causal impulse response, $h(t)$.

- (a) To prove that $H(j\omega)$ is completely specified by $\Re\{H(j\omega)\}$ for a real and causal $h(t)$, we explore the even part of a function, $h_e(t)$.

By definition, $h_e(t) = \frac{1}{2}h(t) + \frac{1}{2}h(-t)$. Since $h(t) = 0$ for $t < 0$,

$$h(t) = \begin{cases} 2h_e(t) & \text{for } t > 0 \\ h_e(t) & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{cases} \quad (1)$$

Therefore, if we know $H_e(j\omega)$, then we can find both $h_e(t)$ and $h(t)$. If $h(t)$ was not causal, we couldn't determine $h(t)$ from $h_e(t)$ alone. We would need $h_o(t)$, the odd part of $h(t)$ also. What is $H_e(j\omega)$? If $h(t)$ is real then $H_e(j\omega) = \Re\{H(j\omega)\}$. This is shown below:

$$\begin{aligned} H_e(j\omega) &= \int_{-\infty}^{\infty} h_e(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{1}{2}h(t)e^{-j\omega t} dt + \int_{-\infty}^{\infty} \frac{1}{2}h(-t)e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2}h(t)e^{-j\omega t} dt + \int_{-\infty}^{\infty} \frac{1}{2}h(t)e^{j\omega t} dt = \int_{-\infty}^{\infty} h(t) \cos(\omega t) dt = \Re\{H(j\omega)\}. \end{aligned}$$

- (b) Given we know $\Re\{H(j\omega)\}$ is $\cos \omega$, we need to find $h(t)$.

$$\Re\{H(j\omega)\} = H_e(j\omega) = \cos \omega = 0.5e^{j\omega} + 0.5e^{-j\omega}$$

$$h_e(t) = \mathcal{F}^{-1}\{\cos(\omega)\} = \mathcal{F}^{-1}\{0.5e^{j\omega}\} + \mathcal{F}^{-1}\{0.5e^{-j\omega}\} = h_{e1}(t) + h_{e2}(t)$$

For $h_{e1}(t)$ we use the time shift property that for any t_o , $e^{-j\omega t_o} X(j\omega) \xleftrightarrow{\mathcal{F}} x(t - t_o)$. Thus for $h_{e1}(t)$, $t_o = -1$. We have

$$h_{e1}(t) = \mathcal{F}^{-1}\{0.5e^{-j\omega \cdot -1} \cdot 1\} = 0.5\delta(t + 1).$$

We find $h_{e2}(t)$ using the same method:

$$h_{e2}(t) = \mathcal{F}^{-1}\{0.5e^{-j\omega \cdot 1} \cdot 1\} = 0.5\delta(t - 1).$$

We combine the two signals to get $h_e(t) = 0.5\delta(t + 1) + 0.5\delta(t - 1)$. Finally, we know because $h(t)$ is causal, our final answer is $h(t) = 2h_e(t) = \delta(t - 1)$.

- (c) We need to show that a real and causal $h(t)$ can be recovered from $h_o(t)$ everywhere except at $t = 0$. By definition, $h_o(t) = \frac{1}{2}h(t) - \frac{1}{2}h(-t)$. Because $h(t)$ is causal, $h_o(t) = \frac{1}{2}h(t)$ for $t > 0$ and $h_o(t) = -\frac{1}{2}h(-t)$ for $t < 0$. This means that if we know $h_o(t)$, we know $h(t) = 2h_o(t)$ when $t > 0$ and $h(t) = 0$ when $t < 0$. However, at $t = 0$ we have a problem. $h_o(0) = 0$ no matter what $h(0)$ is. For example, if $h(t) = \delta(t) + \delta(t - 1)$, then $h_o(t) = \frac{1}{2}\delta(t - 1) + \frac{1}{2}\delta(-t - 1)$. The delta function at $t = 0$ was lost when we looked at the odd part of $h(t)$.

If we do not have a singularity at $t = 0$, but instead has some arbitrary finite value at $t = 0$, then the imaginary part of $H(j\omega)$ can be used to specify $H(j\omega)$. If we have

$$h(t) = \begin{cases} 1 + u(t) & \text{for } t = 0 \\ u(t) & \text{for } t \neq 0 \end{cases} \quad (2)$$

Then $H(j\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t} dt$. The finite value of 1 at $t = 0$ has no area so it doesn't show up under the integral.

$H_o(j\omega) = \frac{1}{2} \int_{-\infty}^{\infty} (h(t) - h(-t))e^{-j\omega t} dt = \frac{1}{2} (\int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt - \int_{-\infty}^{\infty} h(t)e^{j\omega t} dt) = -j \int_{-\infty}^{\infty} h(t) \sin \omega t dt$. This shows that $H_o(j\omega) = \Im\{H(j\omega)\}$. This also shows that $H_o(j\omega) = \frac{1}{2}H(j\omega) - \frac{1}{2}H(-j\omega)$. Thus, $H(j\omega)$ can be recovered from $H_o(j\omega)$. Also the imaginary part can be used to find $h_o(t)$ which can be used to find $h(t)$ everywhere except at $t = 0$.

Problems to be turned in:

Problem 1 Consider the signal $x(t)$ with spectrum depicted in Figure p4.28 (a) of O&W. Sketch the spectrum of

$$y(t) = x(t) [\cos(t/2) + \cos(3t/2)].$$

Solution:

To draw $Y(j\omega)$, the spectrum of $y(t)$, we use the linearity and the multiplication property. Thus, $Y(j\omega) = \frac{1}{2\pi}[X(j\omega) * \mathcal{F}\{\cos \frac{t}{2}\}] + \frac{1}{2\pi}[X(j\omega) * \mathcal{F}\{\cos \frac{3t}{2}\}]$.

$$\mathcal{F}\{\cos \frac{t}{2}\} = \pi[\delta(\omega - 0.5) + \delta(\omega + 0.5)].$$

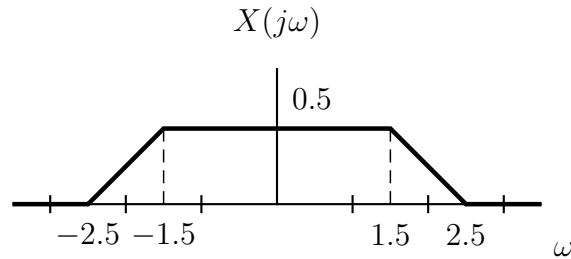
$$\mathcal{F}\{\cos \frac{3t}{2}\} = \pi[\delta(\omega - 1.5) + \delta(\omega + 1.5)].$$

Thus,

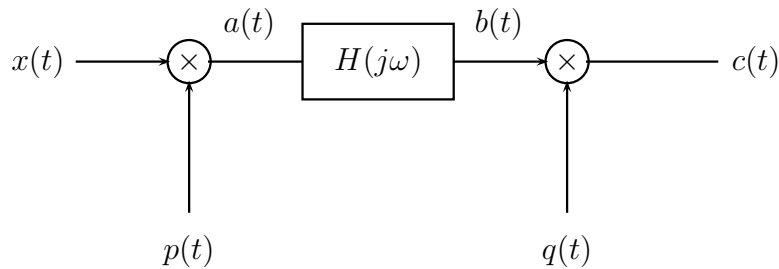
$$Y(j\omega) = \frac{1}{2\pi}X(j\omega) * \pi\delta(\omega - 0.5) + \frac{1}{2\pi}X(j\omega) * \pi\delta(\omega + 0.5)$$

$$\begin{aligned}
& + \frac{1}{2\pi} X(j\omega) * \pi\delta(\omega - 1.5) + \frac{1}{2\pi} X(j\omega) * \pi\delta(\omega + 1.5). \\
& = \frac{1}{2} (X(j(\omega - 0.5)) + X(j(\omega + 0.5)) + X(j(\omega - 1.5)) + X(j(\omega + 1.5)))
\end{aligned}$$

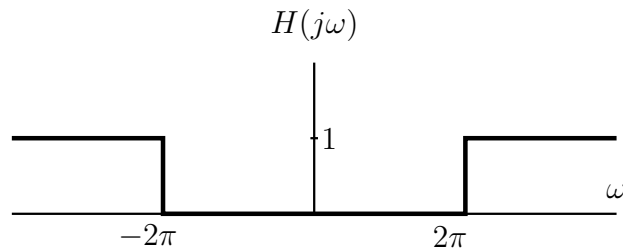
So, $X(j\omega)$ is convolved with 4 shifted impulse functions. Convolution with a shifted impulse function causes the signal to be shifted and replicated about the location on the x-axis where the impulse function is located. Thus, centered at $t = -1.5, -0.5, 0.5,$ and 1.5 , we replicate $X(j\omega)$. We also need to scale these 4 replications by a factor of $\frac{1}{2}$ due to the multiplication of the constants, $\frac{1}{2\pi} \cdot \pi$. This can be seen in the figure below:



Problem 2 Consider the system depicted below:



where $x(t) = \frac{\sin 4\pi t}{\pi t}$, $p(t) = \cos 2\pi t$, $q(t) = \frac{\sin 2\pi t}{\pi t}$, and the frequency response of $H(j\omega)$ is given by

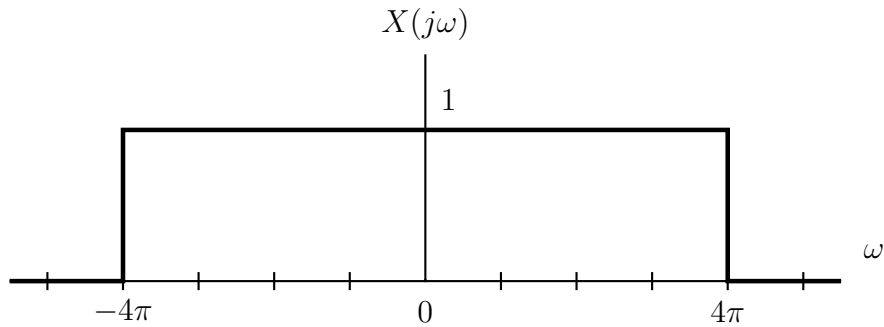


- (a) Let $A(j\omega)$ be the Fourier transform of $a(t)$. Sketch and clearly label $A(j\omega)$.
- (b) Let $B(j\omega)$ be the Fourier transform of $b(t)$. Sketch and clearly label $B(j\omega)$.
- (c) Let $C(j\omega)$ be the Fourier transform of $c(t)$. Sketch and clearly label $C(j\omega)$.
- (d) Compute the output $c(t)$.

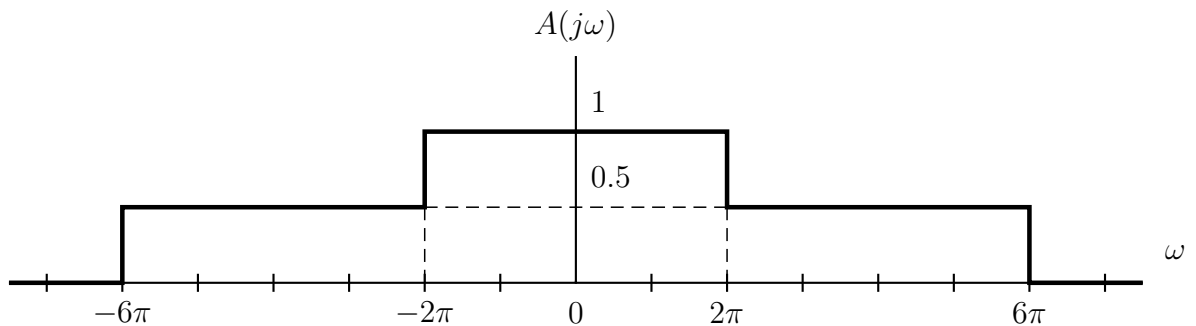
Solution:

- (a) To find $A(j\omega)$, we use the multiplication property. Since $a(t) = x(t) \times p(t)$, then $A(j\omega) = \frac{1}{2\pi}[X(j\omega) * P(j\omega)]$. We need to find $X(j\omega)$ and $P(j\omega)$. To find $X(j\omega)$ from $x(t)$, we recognize $x(t)$ as being in O & W's Table 4.2 Basic Fourier Transform Pairs. It is a sinc function with $W = 4\pi$. Thus,

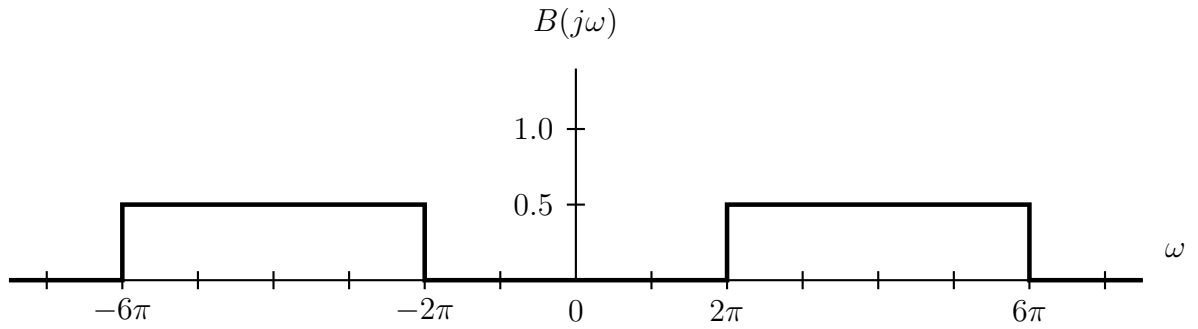
$$X(j\omega) = \begin{cases} 1 & \text{for } |\omega| < 4\pi \\ 0 & \text{for } |\omega| > 4\pi \end{cases} \quad (3)$$



Because $p(t) = \cos 2\pi t$, $P(j\omega) = \pi[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$. Since $P(j\omega)$ is two impulse functions, the convolution of $X(j\omega)$ with $P(j\omega)$ results in the superposition of two copies of $X(j\omega)$, one centered at $\omega = 2\pi$ and the other centered at $\omega = -2\pi$. The resulting $A(j\omega)$ is shown below:



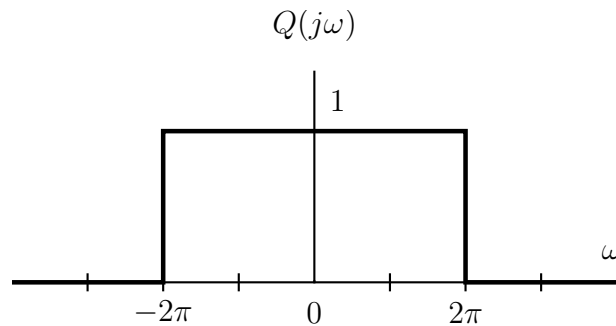
- (b) To find $B(j\omega)$, we use the convolution property. Thus, $B(j\omega) = A(j\omega)H(j\omega)$. $A(j\omega)$ is a low pass filter and $H(j\omega)$ is a high pass filter. Multiplying the two together creates a bandpass filter. $A(j\omega)$ cuts off all frequencies for $|\omega| > 6\pi$. $H(j\omega)$ cuts off all frequencies for $|\omega| < 2\pi$. The resulting signal, $B(j\omega)$ is shown below:



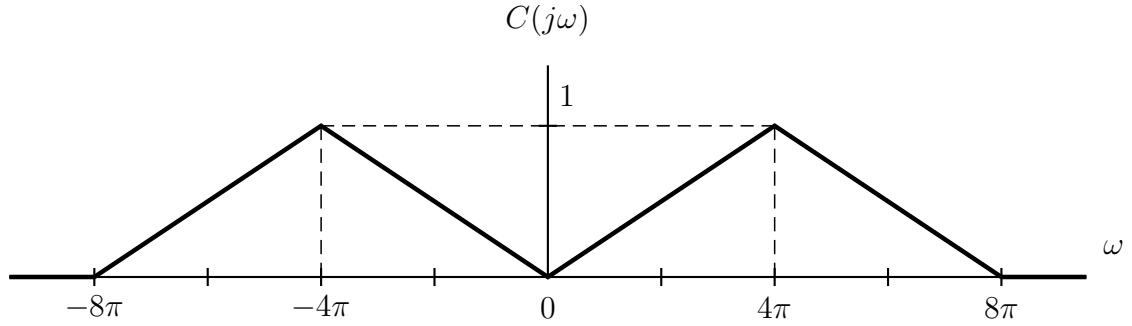
- (c) To find $C(j\omega)$, we need to convolve $B(j\omega)$ with $Q(j\omega)$.

$$Q(j\omega) = \begin{cases} 1 & \text{for } |\omega| < 2\pi \\ 0 & \text{for } |\omega| > 2\pi \end{cases} \quad (4)$$

$Q(j\omega)$ is shown below:



Thus, $C(j\omega)$ can be drawn as shown below:



- (d) To compute $c(t)$, we multiply $b(t)$ with $q(t)$. $B(j\omega)$ is the sum of two ideal frequency-shifted unity-gain filters. Filters in the frequency domain become sinc functions in the time domain. In addition, a frequency shift of ω_o corresponds to multiplying by $e^{-j\omega_o t}$ in the time domain. Hence,

$$b(t) = \frac{1}{2}e^{-j4\pi t} \frac{\sin 2\pi t}{\pi t} + \frac{1}{2}e^{j4\pi t} \frac{\sin 2\pi t}{\pi t} = \cos 4\pi t \frac{\sin 2\pi t}{\pi t}.$$

Therefore

$$c(t) = b(t)q(t) = \cos 4\pi t \frac{\sin^2 2\pi t}{\pi^2 t^2}.$$

Problem 3 O&W 4.44. In addition to parts (a) and (b), answer the following .

- (c) Find the differential equation relating the input and output of this system.

Solution:

- (a) We are given the following equation that relates the output $y(t)$ of a causal LTI system to the input $x(t)$.

$$\frac{dy(t)}{dt} + 10y(t) = \int_{-\infty}^{\infty} x(\tau)z(t-\tau)d\tau - x(t) \quad (5)$$

where

$$z(t) = e^{-t}u(t) + 3\delta(t).$$

We want to find the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ of this system. To do this we take the transform of each term in Equation 5. Because of linearity, to find the Fourier transform of the entire equation, we take the Fourier transform of each term in the equation. For the first term, $\frac{dy(t)}{dt}$, we use the differentiation property to find that

$$\mathcal{F} \left\{ \frac{dy(t)}{dt} \right\} \xleftrightarrow{\mathcal{F}} j\omega Y(j\omega).$$

For the second term, we have $10Y(j\omega)$. For the next term, the integral, we recognize this integral as being the convolution of $x(t)$ and $z(t)$. Thus, the convolution property tells us that

$$\mathcal{F}\{x(t) * z(t)\} \xleftrightarrow{\mathcal{F}} X(j\omega)Z(j\omega).$$

Finally, $x(t)$ becomes $X(j\omega)$. The new equation in the frequency domain is

$$j\omega Y(j\omega) + 10Y(j\omega) = X(j\omega)Z(j\omega) - X(j\omega).$$

Algebraic manipulations are used to separate out $Y(j\omega)$ on the left side of the equation and separate out $X(j\omega)$ on the right side of the equation. We then find

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Z(j\omega) - 1}{10 + j\omega} \quad (6)$$

We insert the function $Z(j\omega)$. By linearity, $\mathcal{F}\{z(t)\} = \mathcal{F}\{e^{-t}u(t)\} + \mathcal{F}\{3\delta(t)\}$. The Fourier transform for each of the terms can be found in O & W's Table 4.2.

$$Z(j\omega) = \frac{1}{1 + j\omega} + 3 = \frac{4 + 3j\omega}{1 + j\omega}$$

This can be plugged into Equation(6) to give

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3 + 2j\omega}{(10 + j\omega)(1 + j\omega)} \quad (7)$$

- (b) The impulse response can be found by doing a partial fraction expansion of $H(j\omega)$ as found in Equation(7).

$$H(j\omega) = \frac{3 + 2j\omega}{(10 + j\omega)(1 + j\omega)} = \frac{A}{10 + j\omega} + \frac{B}{1 + j\omega} = \frac{17/9}{10 + j\omega} + \frac{1/9}{1 + j\omega}.$$

The inverse Fourier transform of the last two terms can be determined from Table 4.2 of O & W to give:

$$h(t) = \left(\frac{1}{9}e^{-t} + \frac{17}{9}e^{-10t} \right) u(t).$$

- (c) To find the differential equation relating the input to the output we go back to Equation 7 and rewrite it as

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3 + 2j\omega}{(j\omega)^2 + 11j\omega + 10}.$$

After cross-multiplying we get

$$10Y(j\omega) + 11j\omega Y(j\omega) + (j\omega)^2 Y(j\omega) = 3X(j\omega) + 2j\omega X(j\omega).$$

Using the differentiation property, we do an inverse Fourier transform to get the differential equation

$$10y(t) + 11\frac{dy(t)}{dt} + 11\frac{d^2y(t)}{dt^2} = 3x(t) + 2\frac{dx}{dt}.$$

Problem 4 O&W 5.21 (c), (g)

Solution:

- (c) We need to compute the Fourier transform of $x[n] = \frac{1}{3}^{|n|} u[-n-2]$. To do this we use the analysis equation as shown below:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{-2} \left(\frac{1}{3}\right)^{-n} e^{-j\omega n} = \sum_{n=2}^{\infty} \left(\frac{1}{3}e^{j\omega}\right)^n = \frac{1}{1 - \frac{1}{3}e^{j\omega}} - 1 - \frac{1}{3}e^{j\omega} = \frac{\frac{1}{9}e^{j2\omega}}{1 - \frac{1}{3}e^{j\omega}}$$

- (g) We need to compute the Fourier transform of $x[n] = \sin(\frac{\pi}{2}n) + \cos(n)$. To do this we use Table 5.2 in O & W to find the Fourier transform of each term and then sum the two transforms for $x[n]$.

$$\mathcal{F}\left\{\sin\left(\frac{\pi}{2}n\right)\right\} = \frac{\pi}{j} \sum_{l=-\infty}^{\infty} (\delta(\omega - \frac{\pi}{2} - 2\pi l) - \delta(\omega + \frac{\pi}{2} - 2\pi l))$$

$$\mathcal{F}\{\cos(n)\} = \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - 1 - 2\pi l) + \delta(\omega + 1 - 2\pi l))$$

$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{l=-\infty}^{\infty} (\delta(\omega - \frac{\pi}{2} - 2\pi l) - \delta(\omega + \frac{\pi}{2} - 2\pi l)) + \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - 1 - 2\pi l) + \delta(\omega + 1 - 2\pi l))$$

Problem 5 The following are Fourier transforms of discrete-time signals. Determine the signal corresponding to each transform.

(a) $X(e^{j\omega}) = 4e^{j4\omega} - e^{j\omega} + 6 + 8e^{-j3\omega} - 16e^{-j11\omega}$

(b)

$$X(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| < \frac{\pi}{4}, \frac{\pi}{2} < |\omega| \leq \pi \\ 0, & \frac{\pi}{4} < |\omega| < \frac{\pi}{2} \end{cases}$$

(c) $X(e^{j\omega}) = \frac{1 + 3e^{-j3\omega}}{1 + \frac{1}{4}e^{-j\omega}}$

Solutions:

- (a) When the Fourier transform is a sum of exponentials, often the easiest way to find $x[n]$ is to use the analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

to match each term to what the particular value of n is. For example, in this problem we can see that the first term, $4e^{j4\omega}$ can only result from the $n = -4$ term. Thus, we can rewrite the equation as

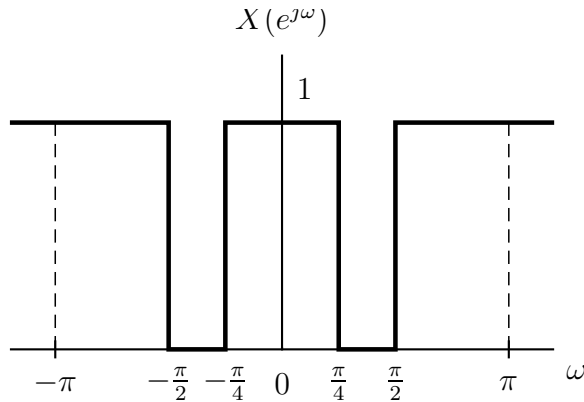
$$X(e^{j\omega}) = x[-4]e^{-j\omega(-4)} + x[-1]e^{-j\omega(-1)} + x[0]e^{-j\omega 0} + x[3]e^{-j\omega(3)} + x[11]e^{-j\omega(11)}.$$

We match terms to find

$$x[-4] = 4, x[-1] = -1, x[0] = 6, x[3] = 8, x[11] = -16.$$

$x[n] = 0$ for all other n .

- (b) $X(e^{j\omega})$ for this problem looks like:



The Fourier transform for this signal as the sum of 3 ideal low pass filters with two of them frequency-shifted to $\omega = \frac{3\pi}{4}$ and $\omega = -\frac{3\pi}{4}$. Table 5.1 in O & W shows that a frequency-shift corresponds to multiplication by an exponential in the time domain:

$$X(e^{j(\omega-\omega_0)}) \xleftrightarrow{\mathcal{F}} e^{j\omega_0 n} x[n].$$

Thus, in the time domain, we can write this as the sum of three sinc functions each with $W = \frac{\pi}{4}$ and with two of them multiplied by the appropriate exponential. This gives

$$x[n] = e^{-j\frac{3\pi}{4}n} \frac{\sin \frac{\pi}{4}n}{\pi n} + \frac{\sin \frac{\pi}{4}n}{\pi n} + e^{j\frac{3\pi}{4}n} \frac{\sin \frac{\pi}{4}n}{\pi n} = \left(2 \cos\left(\frac{3\pi}{4}n\right) + 1 \right) \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}.$$

- (c) This Fourier transform looks similar to the Fourier transform for $a^n u[n]$ in Table 5.2 of O & W. We will manipulate it to look more like that. First, we separate it into two terms so that

$$X(e^{j\omega}) = \frac{1}{1 + \frac{1}{4}e^{-j\omega}} + \frac{3e^{-j3\omega}}{1 + \frac{1}{4}e^{-j\omega}} = X_1(e^{j\omega}) + X_2(e^{j\omega}).$$

By linearity we can solve for the time signal for each transform and then $x[n] = x_1[n] + x_2[n]$. $X_1(e^{j\omega})$ matches to the transform pair mentioned above in Table 5.2 with $a = -\frac{1}{4}$. Thus,

$$x_1[n] = \left(-\frac{1}{4}\right)^n u[n].$$

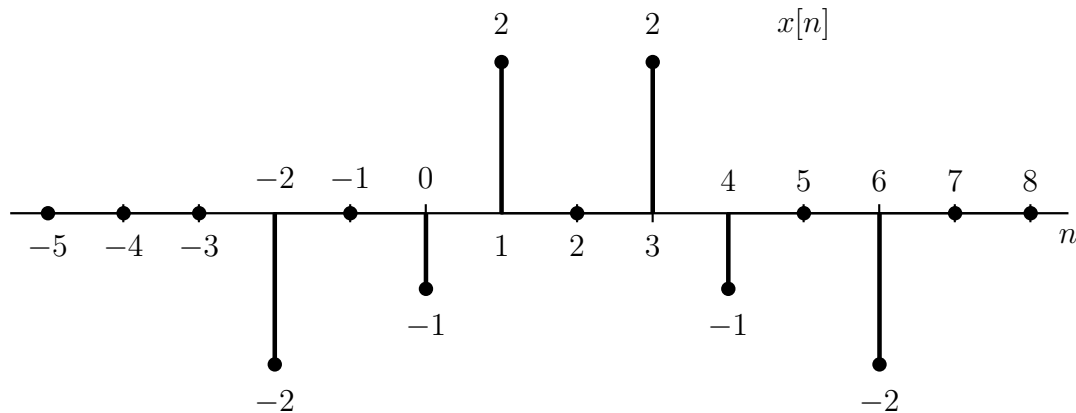
$X_2(e^{j\omega})$ matches to the transform pair in Table 5.2 also except for the numerator term of $3e^{-j3\omega}$. This numerator term corresponds to a scaling of 3 and a time shift of 3. Thus,

$$x_2[n] = 3 \left(-\frac{1}{4}\right)^{n-3} u[n-3].$$

Summing the two terms, we get

$$x[n] = \left(-\frac{1}{4}\right)^n u[n] + 3 \left(-\frac{1}{4}\right)^{n-3} u[n-3].$$

Problem 6 Let $X(e^{j\omega})$ denote the Fourier transform of the signal $x[n]$ depicted below.



- Find $X(1) = X(e^{j0})$.
- Find α such that $e^{j\alpha\omega} X(e^{j\omega})$ is real.
- Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$.
- Find $X(e^{j\pi})$.

(e) Determine and sketch the signal whose Fourier transform is $\Re\{X(e^{j\omega})\}$.

(f) Evaluate each of the following integrals:

$$(f.1) \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$$(f.2) \int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega$$

Solutions:

(a) We use the analysis equation:

$$X(1) = X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]e^{j \cdot 0 \cdot n} = -2.$$

(b) If $x[n]$ is real and even, $X(e^{j\omega})$ is real and even. Since this signal is real, we just need to make it even. If we time shift $x[n]$ 2 steps to the left, it will be real and even. A left time shift of 2 corresponds to multiplying the frequency spectrum by $e^{j\omega 2}$. $\alpha = 2$ will make the frequency spectrum real (and even).

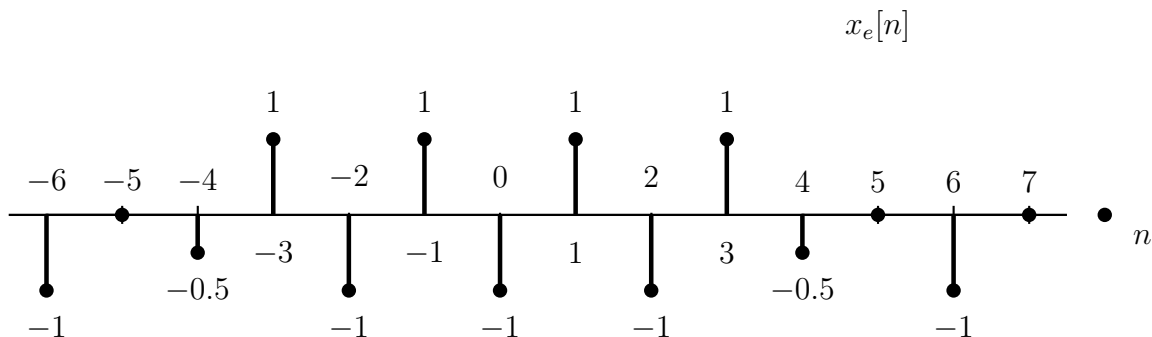
(c) We use the synthesis equation:

$$\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega \cdot 0} d\omega \right) = 2\pi x[0] = -2\pi.$$

(d) We use the analysis equation:

$$X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} = \sum_{n=-\infty}^{\infty} x[n](-1)^n = -10$$

(e) $\Re\{X(e^{j\omega})\}$, which is the real (and even) part of $X(e^{j\omega})$, is the Fourier transform of the even part of the time signal. Thus, we need to sketch $x_e[n] = 0.5x[n] + 0.5x[-n]$. This is shown below:



(f) To evaluate the following integrals we use Parseval's relations.

(f.1) With Parseval's relation, we know that

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

Since we know $x[n]$, we can use it in Parseval's relation to find the quantity we are interested in. Thus,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = 18$$

and $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 18 \cdot 2\pi = 36\pi.$

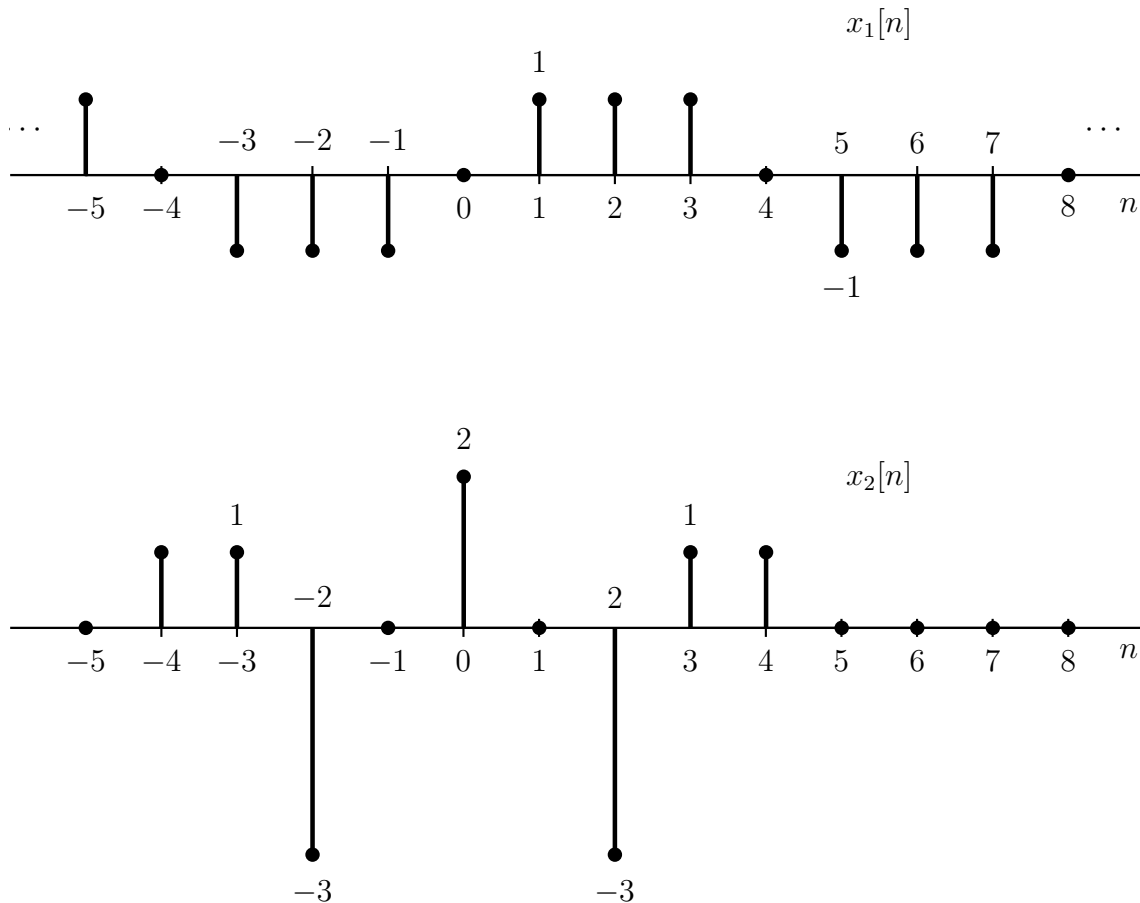
(f.2) To use Parseval's relation again, we need to define a new variable, $Y(e^{j\omega}) = j \frac{dX(e^{j\omega})}{d\omega}$. Then according to Table 5.1 of O&W,

$$\mathcal{F}^{-1} \left\{ j \frac{dX(e^{j\omega})}{d\omega} \right\} = \mathcal{F}^{-1} \{ Y(e^{j\omega}) \} = y[n] = nx[n].$$

Thus,

$$\begin{aligned} \int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega &= \int_{-\pi}^{\pi} \left| j \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = 2\pi \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega \right) \\ &= 2\pi \sum_{n=-\infty}^{\infty} |y[n]|^2 = 2\pi \sum_{n=-\infty}^{\infty} |nx[n]|^2 = 2\pi \cdot 96 = 432\pi. \end{aligned}$$

Problem 7 Answer the questions asked in O&W 5.24 for the following two signals.



Solution:

- (a) (a.1) We want to know if $\Re\{X_1(e^{j\omega})\} = 0$. This is true only if $x_1[n]$ is purely real and odd or purely imaginary and even. Looking at the signal, $x_1[n]$ we see that it is real and odd, or $x_e[n] = 0.5x[n] + 0.5x[-n] = 0$. Yes, $\Re\{X_1(e^{j\omega})\} = 0$.
- (a.2) Because this signal is real and odd, it is purely imaginary. $\Im\{X_1(e^{j\omega})\} \neq 0$.
- (a.3) The real part of a Fourier transform corresponds to the part of the time signal which is real and even and the part of the time signal which is odd and imaginary. By multiplying $X_1(e^{j\omega})$ by $e^{j\alpha\omega}$, we can shift the signal in time. Thus, we can shift by the correct number of units to make the signal even. Looking at the graph of $x_1[n]$ reveals that a time shift of $\alpha = \pm 2$ will cause the signal to be even and thus, the resulting Fourier transform, $e^{j\pm 2\omega} X_1(e^{j\omega})$ will be real.

(a.4) We can use the synthesis equation as follows:

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega \cdot 0}d\omega \right) = 2\pi x[0] = 0.$$

Yes, $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 0$.

(a.5) By definition, the Fourier transform of any discrete sequence is periodic, so $X(e^{j\omega})$ is periodic.

(a.6) Since $x_1[n]$ is periodic,

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta \left(\omega - \frac{2\pi}{N}k \right)$$

where $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-j\frac{2\pi}{N}kn}$. Plugging $\omega = 0$ into the equation above shows that

$$X(e^{j0}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta \left(-\frac{2\pi}{N}k \right).$$

From this equation, we can see that all terms will be zero except possibly $2\pi a_0 \delta(0)$. If $a_0 = 0$ then $X(e^{j0}) = 0$. Using the formula for a_k above we see that,

$$a_0 = \frac{1}{8} \sum_{n=\langle N \rangle} x[n] = 0.$$

We could also note that a_0 is the DC average of the signal and from the graph we can see that the DC average is zero. So, yes, $X(e^{j0}) = 0$.

(b)(b.1) To determine if $\Re\{X(e^{j\omega})\} = 0$, we need to see if $x_2[n]$ is real or imaginary and even or odd. From the graph, we can see that it is real and even. Thus, $X(e^{j\omega})$ is real and even. So $\Re\{X(e^{j\omega})\} \neq 0$.

(b.2) This means that $\Im\{X(e^{j\omega})\} = 0$.

(b.3) Since $X(e^{j\omega})$ is already real, $\alpha = 0$.

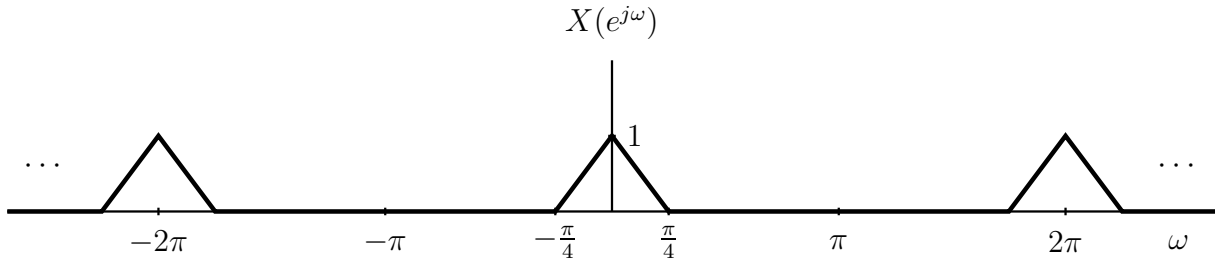
(b.4) Using the same strategy employed for $x_1[n]$, we see that $x_2[0] \neq 0$ so $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega \neq 0$.

(b.5) By definition, all $X(e^{j\omega})$ are periodic so $X_2(e^{j\omega})$ is periodic.

(b.6) $X_2(e^{j0})$ is the DC component of the signal. We can see that the DC component is equal to zero so this condition is met. That is,

$$X_2(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 0.$$

Problem 8 Consider the same question as asked in O&W 5.27 (a) but with $X(e^{j\omega})$ as depicted below



and with

$$p[n] = \cos \pi n - \cos(\pi n/2).$$

Solution:

To sketch $W(e^{j\omega})$, we use the multiplication property,

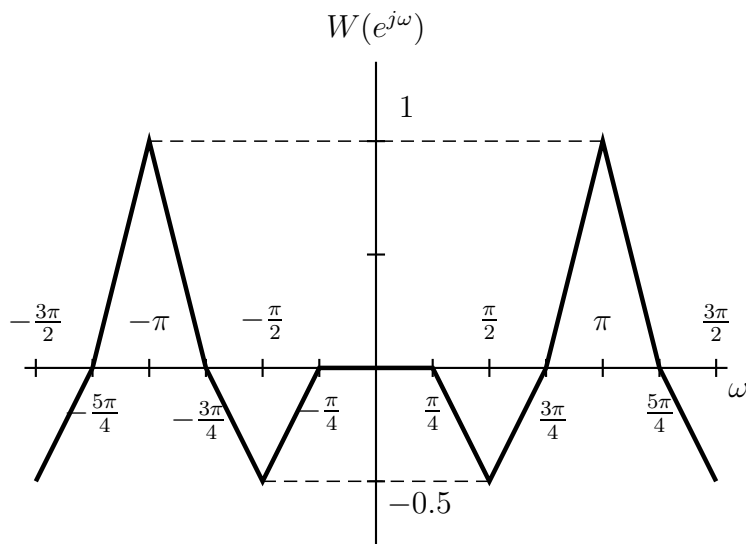
$$p[n]x[n] = w[n] \xleftrightarrow{\mathcal{F}} W(e^{j\omega}) = \frac{1}{2\pi} P(e^{j\omega}) * X(e^{j\omega}).$$

This allows us to use circular convolution to solve the problem. $X(e^{j\omega})$ is as shown above and

$$P(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \pi - 2\pi l) + \delta(\omega + \pi - 2\pi l)) - \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \frac{\pi}{2} - 2\pi l) + \delta(\omega + \frac{\pi}{2} - 2\pi l)).$$

The area under the impulses at $\omega = \pm\pi \pm 2\pi l$ is equal to 2π not just π because the impulses from the term $\sum_{l=-\infty}^{\infty} \delta(\omega - \pi - 2\pi l)$ overlap with the impulses from the term $\sum_{l=-\infty}^{\infty} \delta(\omega + \pi - 2\pi l)$.

These impulses and the impulses at $\omega = \pm\frac{\pi}{2} \pm 2\pi l$ cause replicas of $X(e^{j\omega})$ that are centered at these ω 's. The resulting $W(e^{j\omega})$ is shown below:



Problem 9 Answer the same questions as asked in O&W 5.30 (b) with $x[n]$ as given in that problem and for each of the following LTI unit sample responses:

$$(a) \quad h[n] = \frac{\sin \pi n/16}{\pi n} - \frac{\sin(\pi n/12)}{\pi n}$$

$$(b) \quad h[n] = \frac{\sin(\pi n/8) \sin(\pi n/2)}{\pi^2 n^2}$$

Solution:

We are given that $x[n] = \sin(\frac{\pi n}{8}) - 2 \cos(\frac{\pi n}{4})$.

- (a) To determine $y[n]$, we will use the convolution property. That is, we will compute $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$ and then we can use the synthesis equation to determine $y[n]$ from $Y(e^{j\omega})$. To find $H(e^{j\omega})$ for the given $h[n]$, we use Table 5.1 and Table 5.2 in O&W and find $H(e^{j\omega}) = H_1(e^{j\omega}) - H_2(e^{j\omega})$ where

$$H_1(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \frac{\pi}{16} \\ 0 & \text{for } \frac{\pi}{16} < |\omega| < \pi \end{cases} \quad (8)$$

and

$$H_2(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \frac{\pi}{12} \\ 0 & \text{for } \frac{\pi}{12} < |\omega| < \pi \end{cases} \quad (9)$$

Subtracting $H_2(e^{j\omega})$ from $H_1(e^{j\omega})$ yields

$$H(e^{j\omega}) = \begin{cases} -1 & \text{for } \frac{3\pi}{48} < |\omega| < \frac{4\pi}{48} \\ 0 & \text{for } \frac{3\pi}{48} > |\omega| > \frac{4\pi}{48} \end{cases} \quad (10)$$

$H(e^{j\omega})$ is a bandpass filter whose gain is -1 .

$X(e^{j\omega})$ is found from Table 5.2:

$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{l=-\infty}^{\infty} (\delta(\omega - \frac{\pi}{8} - 2\pi l) - \delta(\omega + \frac{\pi}{8} - 2\pi l)) - 2\pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \frac{\pi}{4} - 2\pi l) + \delta(\omega + \frac{\pi}{4} - 2\pi l)).$$

It is clear that the bandpass filter doesn't allow this $x[n]$ through it and hence $Y(e^{j\omega}) = 0$ and $y[n] = 0$.

- (b) Again we will use the convolution property to find $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$ and then find $y[n]$ from $Y(e^{j\omega})$. $X(e^{j\omega})$ is the same as the previous problem. However,

$$h[n] = \frac{\sin(\frac{\pi n}{8}) \sin \frac{\pi n}{2}}{\pi^2 n^2} = h_1[n]h_2[n].$$

We can use the multiplication property to find $H(e^{j\omega})$. Here,

$$H(e^{j\omega}) = \frac{1}{2\pi} H_1(e^{j\omega}) * H_2(e^{j\omega}).$$

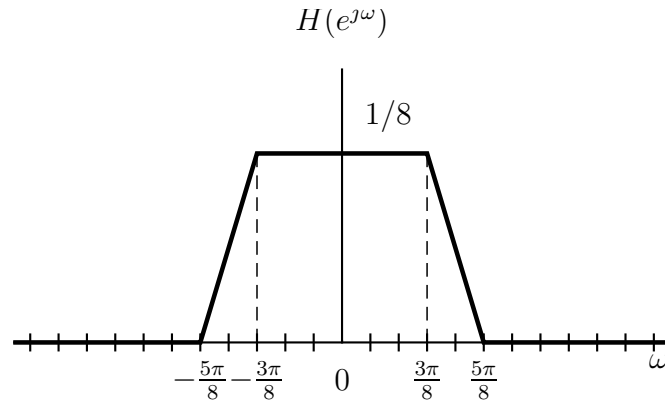
$h_1[n]$ and $h_2[n]$ are sinc functions, thus,

$$H_1(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \frac{\pi}{8} \\ 0 & \text{for } \frac{\pi}{8} < |\omega| < \pi \end{cases} \quad (11)$$

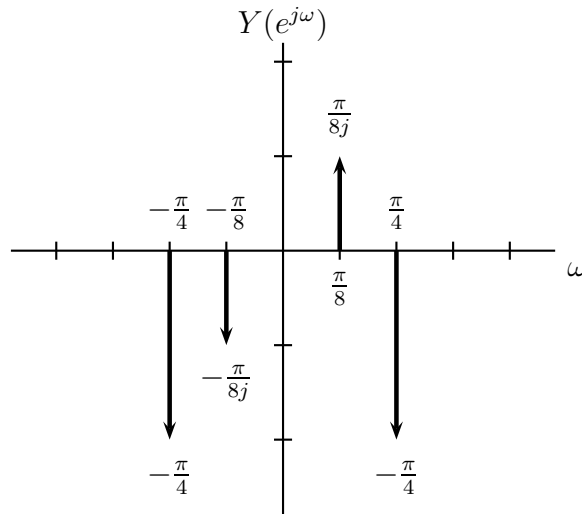
$$H_2(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < |\omega| < \pi \end{cases} \quad (12)$$

The convolution of these two signals is to be calculated.

$H(e^{j\omega})$ is shown below:



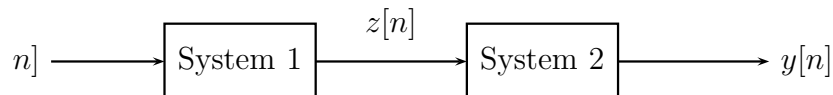
Multiplying $X(e^{j\omega})$ with $H(e^{j\omega})$ gives the following spectrum for $Y(e^{j\omega})$:



The above figure shows that $y[n]$ is just a scaled version of $x[n]$. Thus,

$$y[n] = \frac{1}{8} \sin\left(\frac{\pi n}{8}\right) - \frac{1}{4} \cos\left(\frac{\pi n}{4}\right).$$

Problem 10 Consider a system consisting of the cascade of two LTI systems as depicted below



System 1 is LTI and has a unit-sample response of

$$h[n] = \left(\frac{1}{4}\right)^n u[n].$$

System 2 is LTI, and we know that if the input is

$$x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

the output is

$$y[n] = 10\delta[n] - \delta[n-1].$$

In the following parts, please show your work.

- (a) What is the frequency response $X(e^{j\omega})$ of the overall system ?
- (b) Find the difference equation for the overall system.
- (c) Find the impulse response of the overall system.

Solution:

- (a) For this cascade of two LTI systems,

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}).$$

Taking a Fourier transform of $h_1[n] = (\frac{1}{4})^n u[n]$ gives

$$H_1(e^{j\omega}) = \frac{1}{1 - 0.25e^{-j\omega}}.$$

For $H_2(e^{j\omega})$, we take Fourier transforms of $z[n]$ and $y[n]$ as given above and then $H_2(e^{j\omega}) = \frac{Y(e^{j\omega})}{Z(e^{j\omega})}$. This gives

$$H_2(e^{j\omega}) = \frac{10 - e^{-j\omega}}{1 + 0.5e^{-j\omega}}.$$

Multiplying the two terms gives the overall frequency response as

$$H(e^{j\omega}) = \frac{1}{1 - 0.25e^{-j\omega}} \frac{10 - e^{-j\omega}}{1 + 0.5e^{-j\omega}}.$$

- (b) To find the difference equation for the whole system, we need to multiply $Y(e^{j\omega})$ by the denominator of the above equation and multiply $Z(e^{j\omega})$ by the numerator of the above equation. That is

$$Y(e^{j\omega})(1 + 0.25e^{-j\omega} - 0.125e^{-j2\omega}) = X(e^{j\omega})(10 - e^{-j\omega}).$$

Then,

$$Y(e^{j\omega}) = -0.25e^{-j\omega}Y(e^{j\omega}) + 0.125e^{-j2\omega}Y(e^{j\omega}) + 10X(e^{j\omega}) - 1e^{-j\omega}X(e^{j\omega}).$$

Because of linearity, we take the inverse Fourier transform of each term, noting that multiplication of $e^{-j\omega n_0}$ in the frequency domain corresponds to a shift of $-n_0$ in the time domain. Thus, the difference equation is

$$y[n] = -0.25y[n - 1] + 0.125y[n - 2] + 10x[n] - x[n - 1].$$

- (c) To find the impulse response, $h[n]$, we rewrite the equation for $H(e^{j\omega})$ found in (a) as the sum of two first order systems and do term by term matching to find what the numerator values should be for each term. That is,

$$\begin{aligned} H(e^{j\omega}) &= \frac{10 - e^{-j\omega}}{(1 - 0.25e^{-j\omega})(1 + 0.5e^{-j\omega})} = \frac{A}{(1 - 0.25e^{-j\omega})} + \frac{B}{(1 + 0.5e^{-j\omega})} \\ &= \frac{A(1 + 0.5e^{-j\omega})}{(1 - 0.25e^{-j\omega})(1 + 0.5e^{-j\omega})} + \frac{B(1 - 0.25e^{-j\omega})}{(1 - 0.25e^{-j\omega})(1 + 0.5e^{-j\omega})} \\ &= \frac{A + B + (0.5A - 0.25B)e^{-j\omega}}{(1 - 0.25e^{-j\omega})(1 + 0.5e^{-j\omega})}. \end{aligned}$$

Matching terms gives us $A = 2$ and $B = 8$. Then the inverse transform for each term can be determined from Table 5.2:

$$H(e^{j\omega}) = \frac{2}{(1 - 0.25e^{-j\omega})} + \frac{8}{(1 + 0.5e^{-j\omega})}$$

which gives,

$$h[n] = 2(0.25)^n u[n] + 8(-0.5)^n u[n].$$