Signals and Systems

Fall 2003

Lecture #6

23 September 2003

1. CT Fourier series reprise, properties, and examples
2. DT Fourier series
3. DT Fourier series examples and differences with CTFS

CT Fourier Series Pairs

$$\left(\omega_0 = \frac{2\pi}{T}\right)$$

Review:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

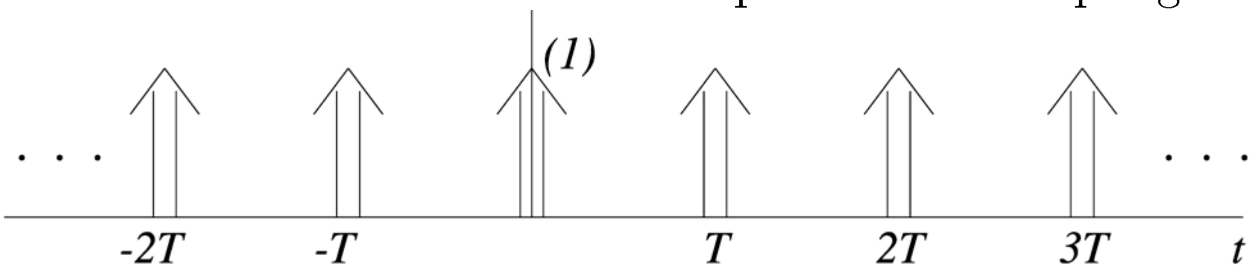
Skip it in future for shorthand

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

Another (important!) example: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad - \quad \text{Sampling function}$$

important for sampling



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T} \quad \text{for } all\ k \ !$$

$$\Downarrow$$

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

— All components have:
(1) the same amplitude,
&
(2) the same phase.

(A few of the) Properties of CT Fourier Series

- Linearity $x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k \;\Rightarrow \alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$
- Conjugate Symmetry

$$x(t) \text{ is real} \quad \Rightarrow \quad a_{-k} = a_k^*$$

$$\Downarrow$$

$$\begin{aligned} a_k &= Re\{a_k\} + jIm\{a_k\} \\ &= |a_k| e^{j\angle a_k} \end{aligned}$$

$$Re\{a_k\} \text{ is even, } Im\{a_k\} \text{ is odd}$$

or

$$|a_k| \text{ is even, } \angle a_k \text{ is odd}$$

- Time shift

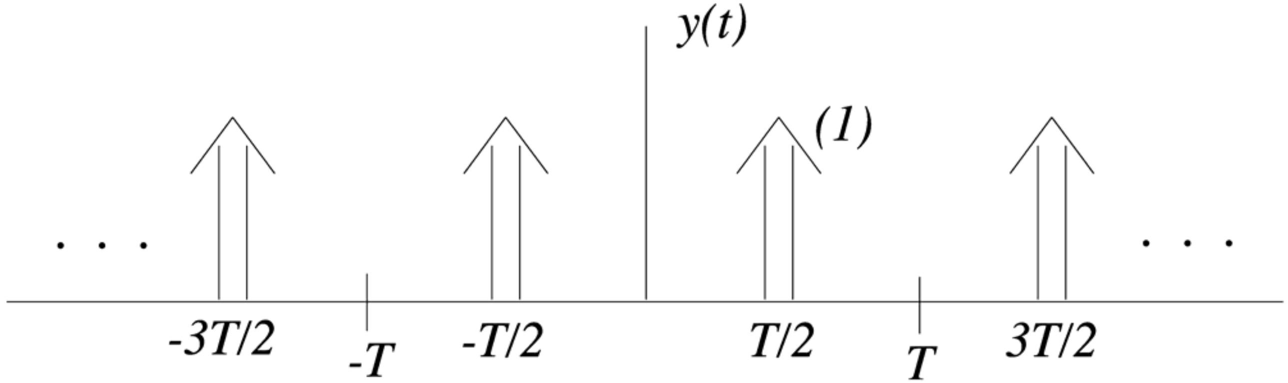
$$\begin{aligned} x(t) &\leftrightarrow a_k \\ x(t-t_0) &\leftrightarrow a_k e^{-jk\omega_0 t_0} = a_k e^{-jk2\pi t_0/T} \end{aligned}$$

Introduces a linear phase shift $\propto t_o$

Example: Shift by half period

$$y(t) = x(t - T/2) \;\leftrightarrow\; a_k e^{-jk\pi} = (-1)^k a_k$$

$$\text{using} \quad e^{-jk\omega_0 T/2} \;=\; e^{-jk\pi}$$



$$y(t) \leftrightarrow \quad (-1)^k a_k \quad \left(a_k = \frac{1}{T} = \text{ F.C. of } \sum_{n=-\infty}^{\infty} \delta(t - nT) \right)$$

$$\|$$

$$\frac{(-1)^k}{T}$$

- **Parseval's Relation**

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2 dt}_{\text{Average signal power}} = \sum_{k=-\infty}^{\infty} \underbrace{|a_k|^2}_{\text{Power in the } k_{th} \text{ harmonic}}$$

Energy is the same whether measured in the time-domain or the frequency-domain

- **Multiplication Property**

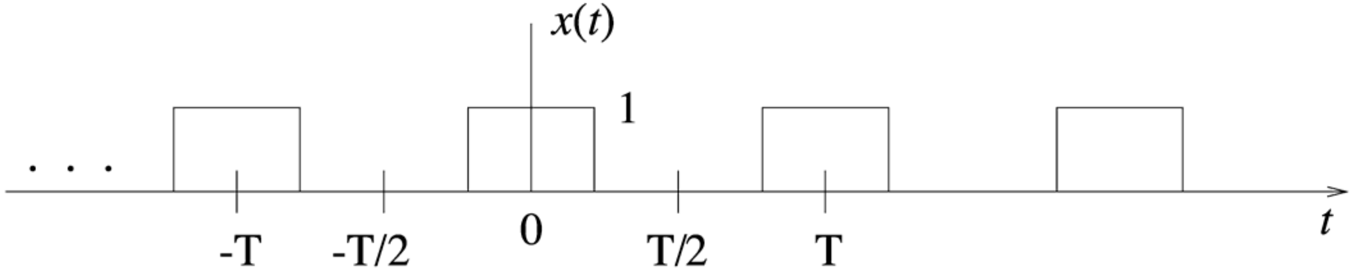
$$\begin{aligned} x(t) \leftrightarrow a_k, y(t) &\leftrightarrow b_k \quad \text{(Both } x(t) \text{ and } y(t) \text{ are} \\ &\Downarrow \qquad \text{periodic with the same period } T) \\ x(t)\cdot y(t) &\leftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k \end{aligned}$$

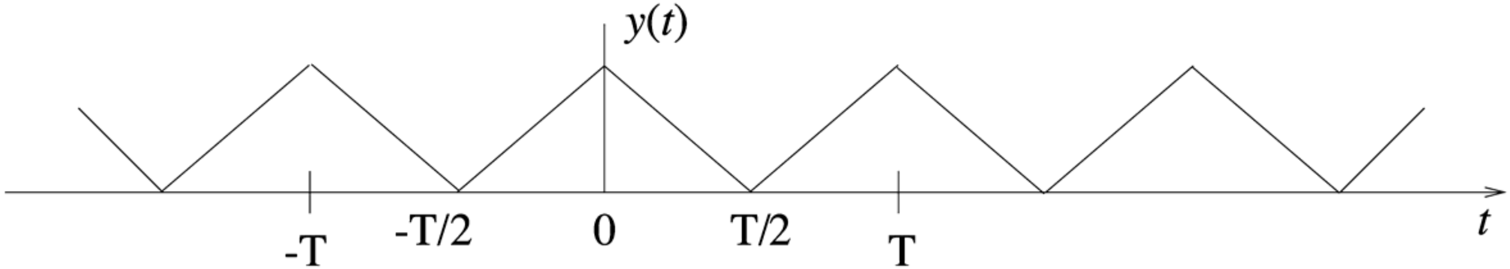
Proof:

$$\underbrace{\sum_l a_l e^{jl\omega_0 t}}_{x(t)} \cdot \underbrace{\sum_m b_m e^{jm\omega_0 t}}_{y(t)} = \sum_{l,m} a_l b_m e^{j(l+m)\omega_0 t} \xrightarrow{l+m=k} \sum_k \underbrace{\left[\sum_l a_l b_{k-l}\right]}_{c_k} e^{jk\omega_0 t}$$

Periodic Convolution

$x(t)$, $y(t)$ periodic with period T





$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \quad - \text{ not very meaningful}$$

E.g. If both $x(t)$ and $y(t)$ are positive, then

$$x(t) * y(t) = \infty$$

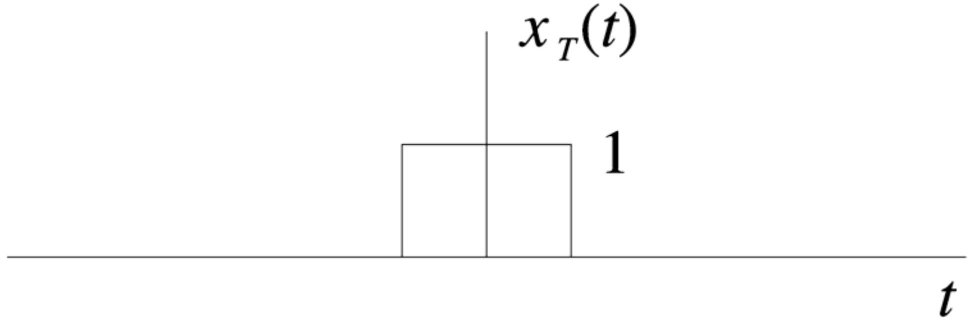
Periodic Convolution (continued)

Periodic convolution: Integrate over *any* one period (e.g. -T/2 to T/2)

$$z(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \;= \int_{-\infty}^{\infty} x_T(\tau)y(t-\tau)d\tau$$

where

$$x_T(t) = \begin{cases} x(t) & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$



Periodic Convolution (continued) Facts

1) $z(t)$ is periodic with period T (why?)

From Lecture #2, $x(t) = x(t+T) \rightarrow y(t) = y(t+T)$ for LTI systems.

In the convolution, treat $y(t)$ as the input and $x_T(t)$ as $h(t)$

2) Doesn't matter what period over which we choose to integrate:

$$z(t) = \int_T x(\tau) y(t-\tau) d\tau = x(t) \otimes y(t)$$

Periodic convolution in time

3) $$x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k, z(t) \leftrightarrow c_k$$

$$\begin{aligned} c_k &= \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T \left(\int_T x(\tau) y(t-\tau) d\tau\right) e^{-jk\omega_0 t} dt \\ &= \int_T \underbrace{\left(\frac{1}{T}\int_T y(t-\tau) e^{-jk\omega_0 (t-\tau)} dt\right)}_{b_k} x(\tau) e^{-jk\omega_0 \tau} d\tau \\ &= \int_T b_k x(\tau) e^{-jk\omega_0 \tau} d\tau = T a_k b_k \end{aligned}$$

Multiplication in frequency!

Fourier Series Representation of DT Periodic Signals

- $x[n]$ - periodic with fundamental period N, fundamental frequency

$$x[n+N] = x[n] \quad \text{and} \quad \omega_0 = \frac{2\pi}{N}$$

- Only $e^{j\omega n}$ which are periodic with period N will appear in the *FS*

$$\omega N = k2\pi \Leftrightarrow \omega = k\omega_0 \quad , \quad k = 0, \pm 1, \pm 2, \cdots$$

- There are only N distinct signals of this form

$$e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} e^{j\overbrace{N\omega_0 n}^{2\pi n}} = e^{jk\omega_0 n}$$

- So we *could* just use $e^{j0\omega_0 n}, e^{j1\omega_0 n}, e^{j2\omega_0 n}, \cdots, e^{j(N-1)\omega_0 n}$
- However, it is often useful to allow the choice of N consecutive values of k to be *arbitrary*.

$$\Downarrow$$

DT Fourier Series Representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$\sum_{k=<N>}$ = Sum over *any* N consecutive values of k

— This is a *finite* series

$\{a_k\}$ - Fourier (series) coefficients

Questions:

1) What DT periodic signals have such a representation?

2) How do we find a_k?

Answer to Question #1:

Any DT periodic signal has a Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\Downarrow$$

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jk\omega_0}$$

$$x[2] = \sum_{k=<N>} a_k e^{j2k\omega_0}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{j(N-1)k\omega_0}$$

N equations for N unknowns, $a_0, a_1, \cdots, a_{N-1}$

A More Direct Way to Solve for a_k

Finite geometric series

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & ,\alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & ,\alpha \neq 1 \end{cases}$$

$$\Downarrow \qquad \alpha = e^{jk\omega_0}$$

$$\sum_{n=<N>} e^{jk\omega_0 n} = \sum_{n=0}^{N-1} \left(e^{jk\omega_0}\right)^n = \sum_{n=0}^{N-1} \left(e^{jk2\pi/N}\right)^n$$

$$= \begin{cases} N & ,k = 0, \pm N, \pm 2N, \cdots \\ \dfrac{1-e^{jk(2\pi/N)N}}{1-e^{jk\omega_0}} = 0 & ,\text{otherwise} \end{cases}$$

So, from $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$

$\Downarrow$ multiply both sides by $e^{-jm\omega_0 n}$ and then $\sum_{n=<N>}$

$$\begin{aligned}\sum_{n=<N>} x[n]e^{-jm\omega_0 n} &= \sum_{n=<N>} \left(\sum_{k=<N>} a_k e^{jk\omega_0 n} \right) e^{-jm\omega_0 n} \\ &= \sum_{k=<N>} a_k \underbrace{\left(\sum_{n=<N>} e^{j(k-m)\omega_0 n} \right)}_{=N\delta[k-m] - \text{ orthogonality}} \\ &= N a_m \\ &\Downarrow\end{aligned}$$

DT Fourier Series Pair $\left(\omega_0 = \frac{2\pi}{N}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equaiton)}$$

Note: It is convenient to think of a_k as being defined for *all* integers k. So:

1) $a_{k+N} = a_k$ — Special property of DT Fourier Coefficients.

2) We only use N consecutive values of a_k in the synthesis equation. (Since $x[n]$ is periodic, it is specified by N numbers, either in the time or frequency domain)

Example #1: Sum of a pair of sinusoids

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

$$- \text{ periodic with period } N = 16 \Rightarrow \omega_0 = \pi/8$$

$$x[n] = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] + \frac{1}{2}\left[e^{j\pi/4}e^{j2\omega_0 n} + e^{-j\pi/4}e^{-j2\omega_0 n}\right]$$

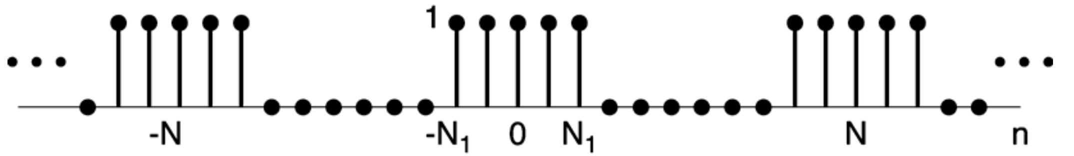
$$\Downarrow$$

$$a_0 = 0$$
$$a_1 = 1/2$$
$$a_{-1} = 1/2$$
$$a_2 = e^{j\pi/4}/2$$
$$a_{-2} = e^{-j\pi/4}/2$$
$$a_3 = 0$$
$$a_{-3} = 0$$
$$\vdots$$

$$a_{15} = a_{-1+16} = a_{-1} = 1/2$$

$$a_{66} = a_{2+4\times 16} = a_2 = e^{j\pi/4}/2$$

Example #2: DT Square Wave



$$a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] = \frac{2N_1+1}{N} = a_N = a_{-N} = a_{6N} = \cdots$$

For $k \neq$ multiple of N:

Using $n = m - N_1$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\omega_0 (m-N_1)}$$

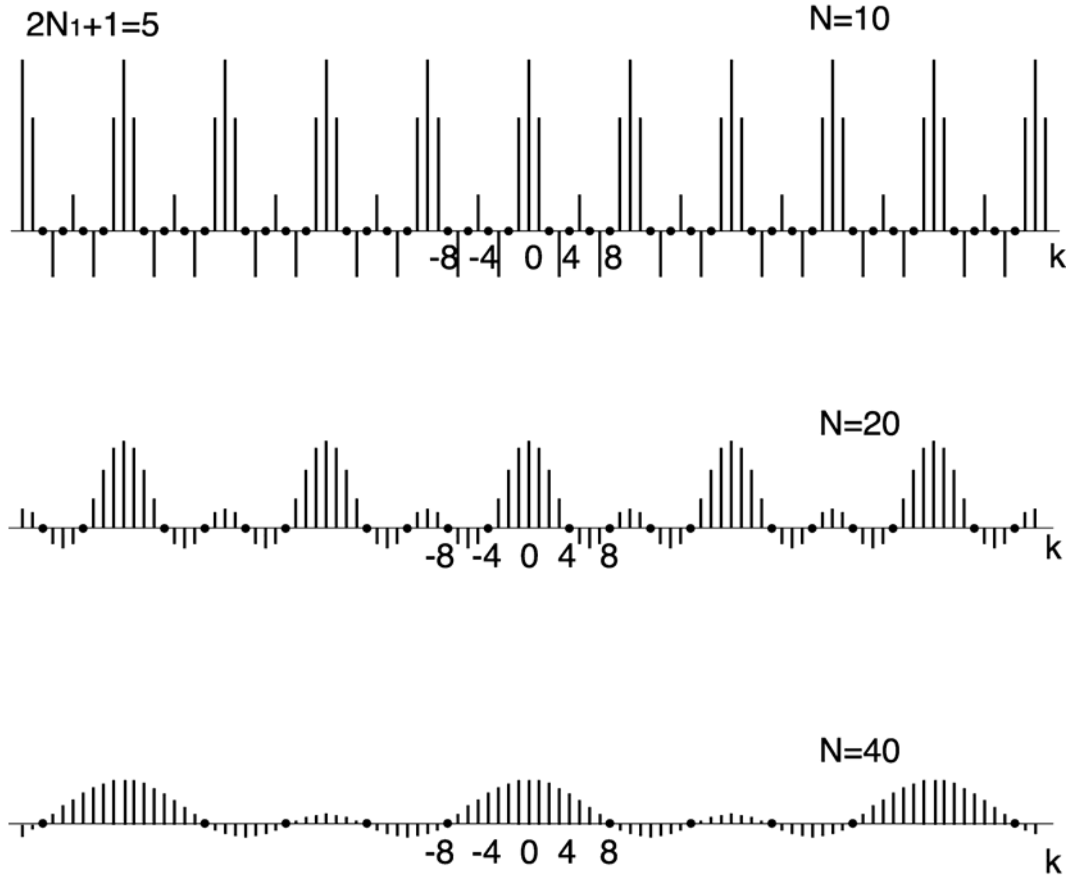
$$= \frac{1}{N} e^{jk\omega_0 N_1} \sum_{m=0}^{2N_1} (e^{-jk\omega_0})^m = \frac{1}{N} e^{jk\omega_0 N_1} \frac{1 - e^{-jk\omega_0 (2N_1+1)}}{1 - e^{jk\omega_0}}$$

$$= \frac{1}{N} \frac{\sin[k(N_1+1/2)\omega_0]}{\sin(k\omega_0/2)} = \frac{1}{N} \frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}$$

$$\Downarrow$$

Example #2: DT Square wave (continued)

$$a_k = \frac{1}{N} \frac{\sin\left[2\pi k(N_1 + 1/2)/N\right]}{\sin(\pi k/N)}$$



Convergence Issues for DT Fourier Series:

Not an issue, since all series are finite sums.

Properties of DT Fourier Series: Lots, just as with CT Fourier Series

Example:

$$x[n] \leftrightarrow a_k$$
$$e^{jM\omega_0 n}x[n] \leftrightarrow b_k = ?$$

$$x[n]e^{jM\omega_0 n} = \sum_{r=<N>} a_r e^{jr\omega_0 n} e^{jM\omega_0 n}$$
$$\overset{k=r+M}{=} \sum_{k=<N>} a_{k-M} e^{jk\omega_0 n}$$

$$\Downarrow$$

$$b_k = a_{k-M}$$

Frequency shift

$$jk\omega_0 \to j(k-M)\omega_0$$