Signals and Systems

Fall 2003

Lecture #11

9 October 2003

1. DTFT Properties and Examples
2. Duality in FS & FT
3. Magnitude/Phase of Transforms and Frequency Responses

Convolution Property Example

$$h[n] = \alpha^n u[n], \quad x[n] = \beta^n u[n] \qquad |\alpha|, |\beta| < 1$$

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}, \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

$$\Downarrow$$

$$y[n] = h[n] * x[n] \longleftrightarrow Y(e^{j\omega}) = \left(\frac{1}{1 - \alpha e^{-j\omega}}\right)\left(\frac{1}{1 - \beta e^{-j\omega}}\right)$$

- ratio of polynomials in $e^{-j\omega}$

$$\beta \neq \alpha : \; Y(e^{j\omega}) \overset{\text{PFE}}{=} \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}}$$

A, B - determined by partial fraction expansion

$$y[n] = A\alpha^n u[n] + B\beta^n u[n]$$

$$\beta = \alpha : \; Y(e^{j\omega}) = \left(\frac{1}{1 - \alpha e^{-j\omega}}\right)^2$$

$$y[n] \quad = \underset{nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}}{\longrightarrow} \qquad y[n] = (n+1)\alpha^n u[n]$$

DT LTI System Described by LCCDE's

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

From time-shifting property: $x[n-k] \longleftrightarrow e^{-j\omega k} X(e^{j\omega})$

$$\Downarrow$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\Downarrow$$

$$Y(e^{j\omega}) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}} \right]}_{H(e^{j\omega})} X(e^{j\omega})$$

— Rational function of $e^{-j\omega}$, use PFE to get $h[n]$

Example: First-order recursive system

$$y[n] - \alpha y[n-1] = x[n], \quad |\alpha| < 1$$

with the condition of initial rest $\Leftrightarrow$ *causal*

$$(1 - \alpha e^{-j\omega})Y(e^{j\omega}) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \alpha e^{-j\omega}}$$

$$\Downarrow$$

$$h[n] = \alpha^n u[n]$$

DTFT Multiplication Property

$$y[n] = x_1[n] \cdot x_2[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega})$$

$\hookrightarrow$ Periodic Convolution

Derivation:

$$\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_1[n] \cdot x_2[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta}) e^{j\theta n} d\theta \right) x_2[n] e^{-j\omega n} \\
&= \frac{1}{2\pi}\int_{2\pi} (X_1(e^{j\theta}) \underbrace{\sum_{n=-\infty}^{\infty} x_2[n] e^{-j(\omega-\theta)n}}_{X_2(e^{j(\omega-\theta)})}) d\theta \\
&= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta
\end{aligned}$$

Calculating Periodic Convolutions

Suppose we integrate from $-\pi$ to π:

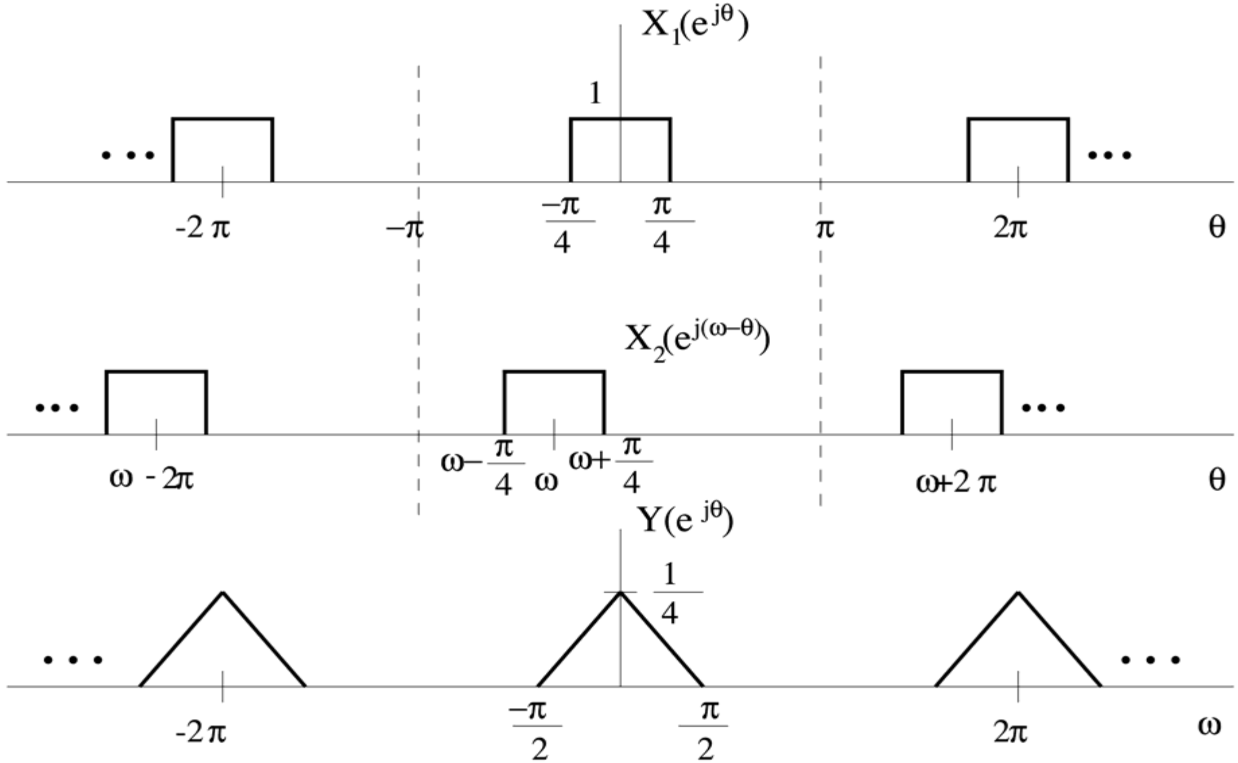
$$
\begin{aligned}
Y(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta
\end{aligned}
$$

where

$$
\hat{X}_1(e^{j\theta}) = \begin{cases} X_1(e^{j\theta}), & |\theta| \le \pi \\ 0, & \text{otherwise} \end{cases}
$$

Example: $y[n] = \left(\frac{\sin(\pi n/4)}{\pi n}\right)^2 = x_1[n] \cdot x_2[n], \quad x_1[n] = x_2[n] = \frac{\sin(\pi n/4)}{\pi n}$

$$Y(e^{j\omega}) = \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega})$$



Duality in Fourier Analysis

Fourier Transform is highly symmetric

CTFT: *Both* time and frequency are continuous and in general *aperiodic*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Same except for these differences

$$\Downarrow$$

Suppose $f(\cdot)$ and $g(\cdot)$ are two functions related by

$$f(r) = \int_{-\infty}^{\infty} g(\tau)e^{-jr\tau}\,d\tau$$

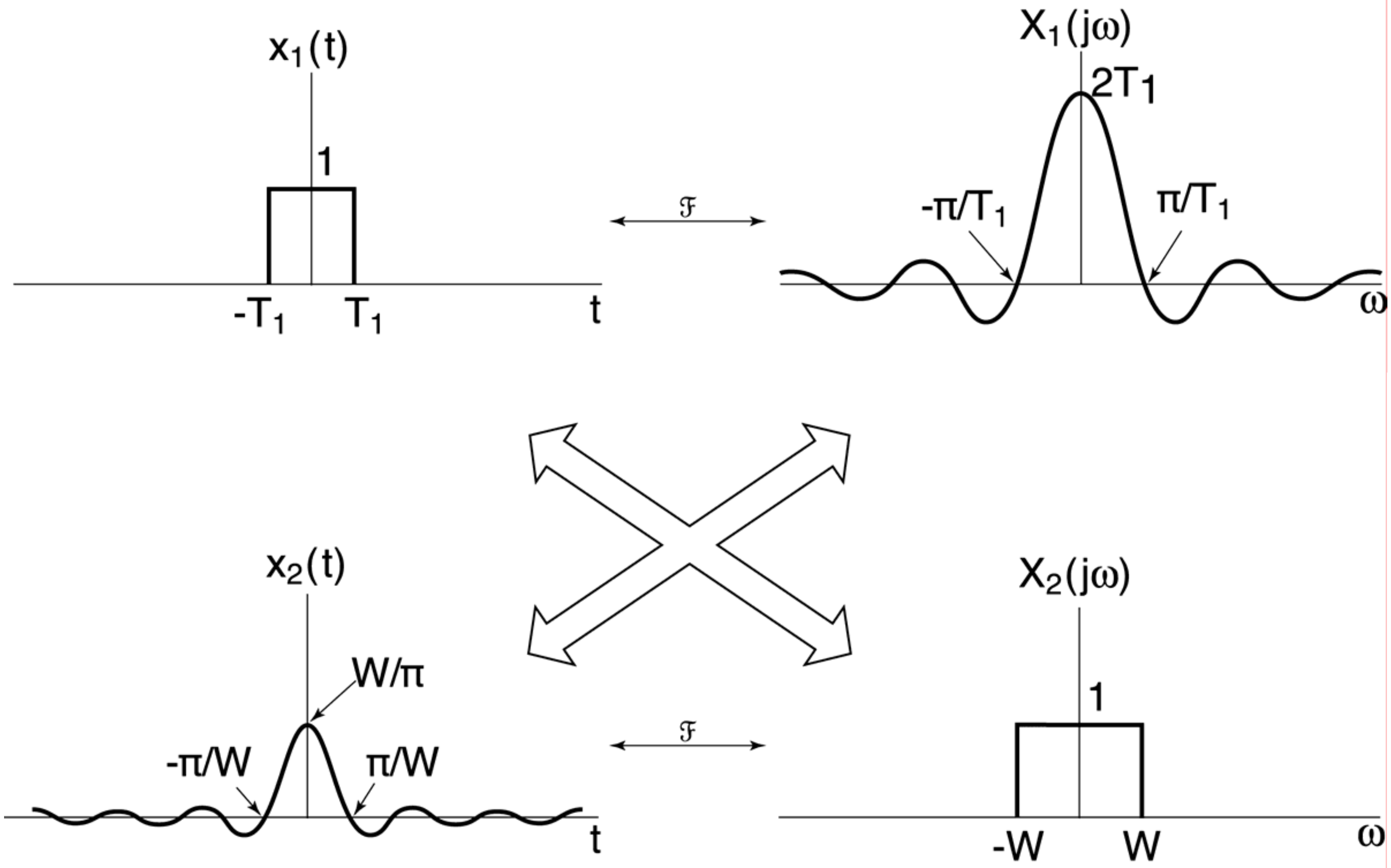
Then

Let $\tau = t$ and $r = \omega$: $\quad x_1(t) = g(t) \longleftrightarrow X_1(j\omega) = f(\omega)$

Let $\tau = -\omega$ and $r = t$: $\quad x_2(t) = f(t) \longleftrightarrow X_2(j\omega) = 2\pi g(-\omega)$

Example of CTFT duality

Square pulse in either time or frequency domain



DTFS

Discrete & periodic in time $\longleftrightarrow$ *Periodic & discrete* in frequency

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = x[n+N], \quad \omega_0 = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N} \sum_{k=<N>} x[n] e^{-jk\omega_0 n} = a_{k+N}$$

Duality in DTFS

Suppose $f[\cdot]$ and $g[\cdot]$ are two functions related by

$$f[m] = \frac{1}{N} \sum_{r=<N>} g[r] e^{-jr\omega_0 m}$$

$$\Rightarrow g[r] = \sum_{m=<N>} f[m] e^{jr\omega_0 m}$$

Then

Let $m = n$ and $r = -k$: $\quad x_1[n] = f[n] \longleftrightarrow a_k = \frac{1}{N} g[-k]$

Let $r = n$ and $m = k$: $\quad x_2[n] = g[n] \longleftrightarrow a_k = f[k]$

Duality between CTFS and DTFT

CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = x(t+T), \quad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Periodic in time $\leftrightarrow$ *Discrete* in frequency

DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X\left(e^{j(\omega+2\pi)}\right)$$

Discrete in time $\leftrightarrow$ *Periodic* in frequency

CTFS-DTFT Duality

Suppose $f(\cdot)$ is a CT signal and $g[\cdot]$ a DT sequence related by

$$f(\tau) = \sum_{m=-\infty}^{+\infty} g[m] e^{jm\tau} = f(\tau + 2\pi)$$

Then

$$x(t) = f(t) \longleftrightarrow a_k = g[k]$$
$$\text{(periodic with period } 2\pi\text{)}$$

$$x[n] = g[n] \longleftrightarrow X(e^{j\omega}) = f(-\omega)$$

Magnitude and Phase of FT, and Parseval Relation

CT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$$

Parseval Relation:

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}\frac{1}{2\pi}\underbrace{|X(j\omega)|^2}_{\text{Energy density in }\omega}d\omega$$

Energy density in ω

DT:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

Parseval Relation:

$$\sum_{n=-\infty}^{\infty}|x[n]|^2 = \int_{2\pi}\frac{1}{2\pi}|X(e^{j\omega})|^2 d\omega$$

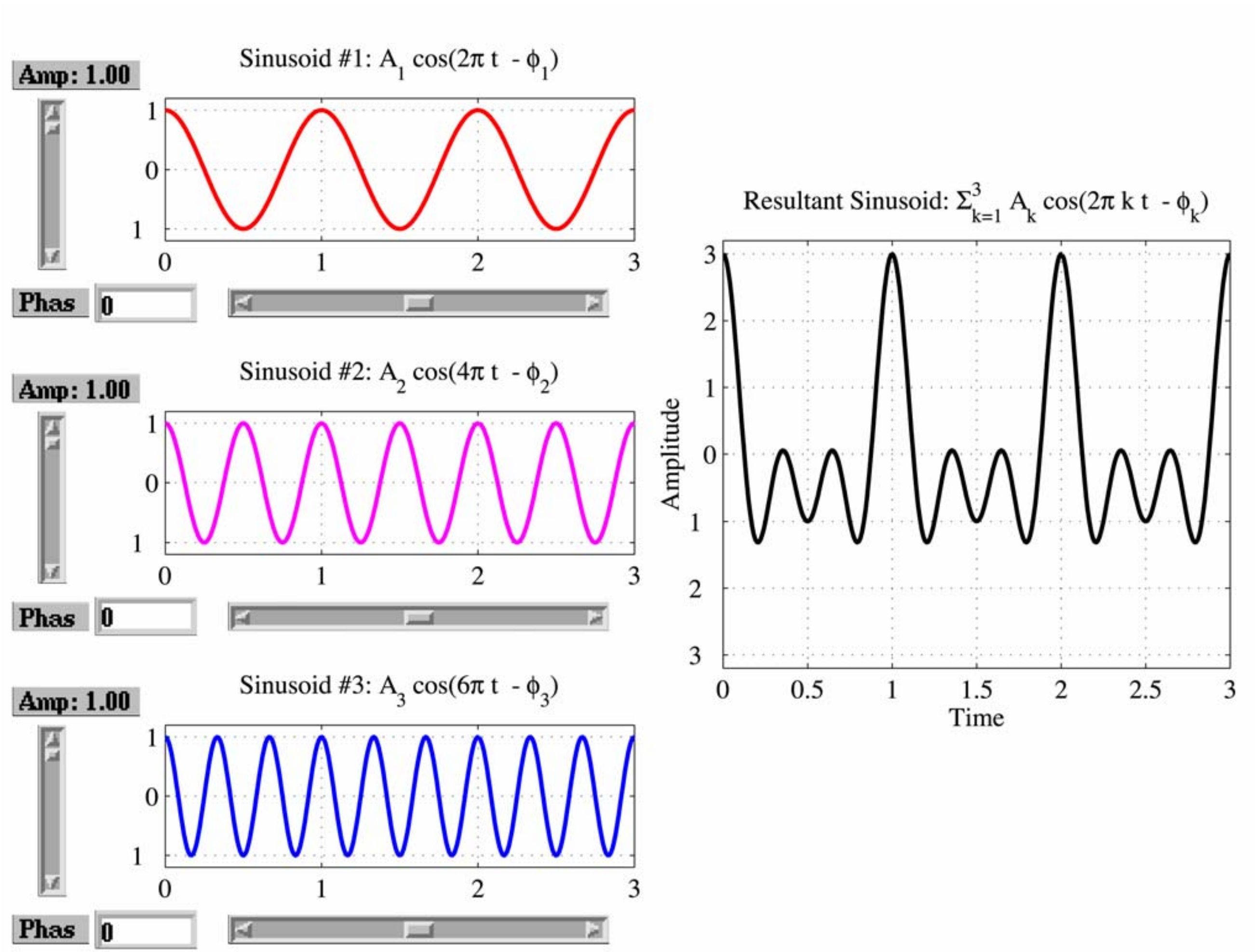
Effects of Phase

- *Not* on signal energy distribution as a function of frequency

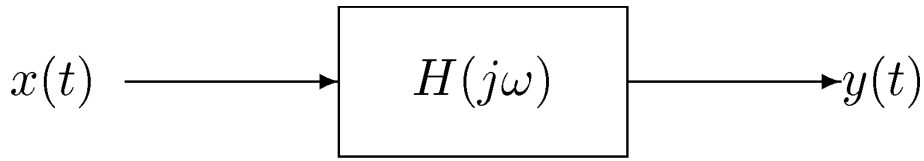
- *Can* have dramatic effect on signal shape/character
  - Constructive/Destructive interference

- Is that important?
  - Depends on the signal and the context

Demo: 1) Effect of phase on Fourier Series
2) Effect of phase on image processing



Log-Magnitude and Phase



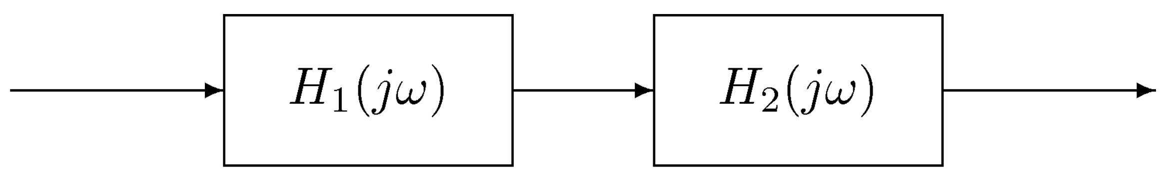
$$\Downarrow$$

$$|Y(j\omega)| = |H(j\omega)| \cdot |X(j\omega)|$$

or $$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$$

and $$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Cascading:



$$\left.\begin{aligned} \log|H(j\omega)| &= \log|H_1(j\omega)| + \log|H_2(j\omega)| \\ \angle H(j\omega) &= \angle H_1(j\omega) + \angle H_2(j\omega) \end{aligned}\right\} \text{Easy to add}$$

Plotting Log-Magnitude and Phase

a) For real-valued signals and systems

$$\left.\begin{array}{rcl} |H(-j\omega)| & = & |H(j\omega)| \\ \angle H(-j\omega) & = & -\angle H(j\omega) \end{array}\right\} \Rightarrow$$

Plot for $\omega \geq 0$, often with a *logarithmic* scale for frequency in CT

b) In DT, need only plot for $0 \leq \omega \leq \pi$ (with *linear* scale)

c) For historical reasons, log-magnitude is usually plotted in units of *decibels* (dB): $\left(1 \; bel = 10 \; decibels = \frac{\text{output power}}{\text{input power}} = 10\right)$

power ↙ magnitude ↙

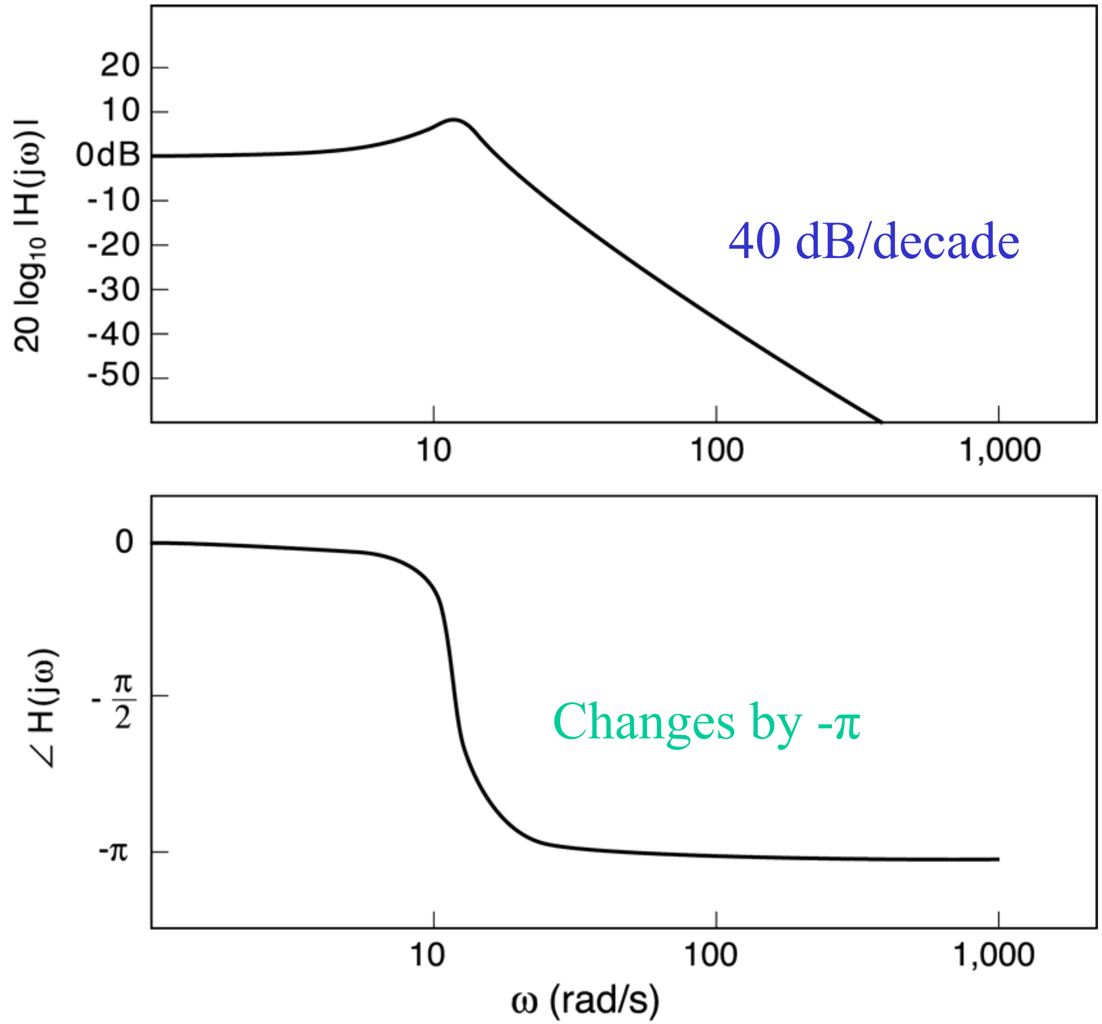
$$10\log_{10}|H(j\omega)|^2 = 20\log_{10}|H(j\omega)|$$

$$\begin{array}{lcr} |H(j\omega)| = 1 & \longrightarrow & 0\,\text{dB} \\ |H(j\omega)| = \sqrt{2} & \longrightarrow & \sim 3\,\text{dB} \\ |H(j\omega)| = 2 & \longrightarrow & \sim 6\,\text{dB} \\ |H(j\omega)| = 10 & \longrightarrow & 20\,\text{dB} \\ |H(j\omega)| = 100 & \longrightarrow & 40\,\text{dB} \end{array}$$

So… 20 dB or 2 *bels*:
= 10 amplitude gain
= 100 power gain

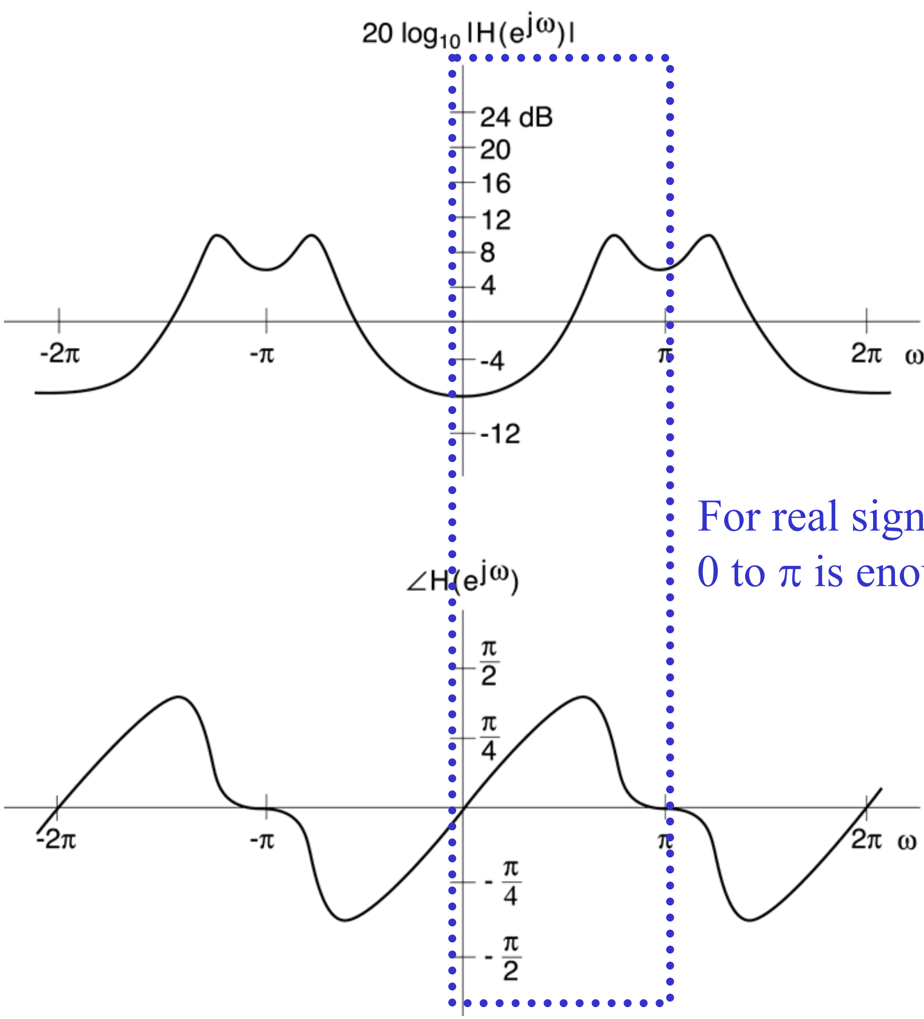
A Typical Bode plot for a second-order CT system

20 log$|H(j\omega)|$ and $\angle H(j\omega)$ vs. log ω



A typical plot of the magnitude and phase of a second-order DT frequency response

$20\log|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ vs. ω



For real signals, 0 to π is enough