MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Fall 2003

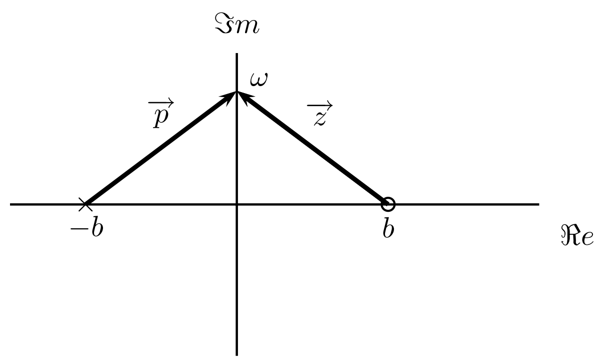
PROBLEM SET 9 SOLUTIONS

Home Study 1 O&W 9.25(e)
We assume that the system we are considering, $H(s)$ takes the following form:

$$\begin{aligned} H(s) &= \frac{s-b}{s+b}(s+a+j\omega_0)(s+a-j\omega_0) \\ &= \frac{s-b}{s+b}(s^2+2as+a^2+\omega_0^2), \end{aligned}$$

where b corresponds to the pole and the zero on the real axis and a and ω_0 are characterizing the complex conjugate zeros with $\omega_0 > a$. Thus, we can see that $H(s)$ is a cascade of two LTI systems, $H_1(s)$ and $H_2(s)$ each of which is examined more detailed below.

The pole-zero diagram of $H_1(s)$ is shown below. It has a pole and a zero on the real axis which are spaced equally from the origin.



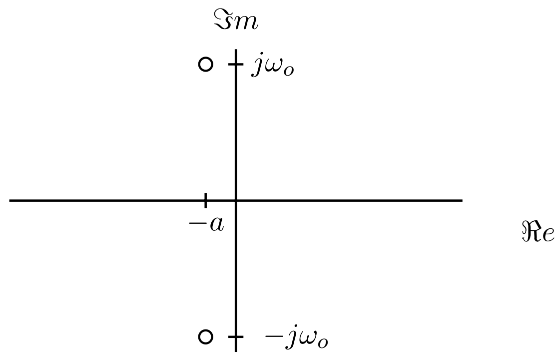
The magnitude response is the ratio of the magnitude of the vectors from the zeros to the vectors from the poles as we traverse the $j\omega$ axis. From any point along the $j\omega$-axis, the pole and zero vectors have equal length, and, consequently, the magnitude of the frequency response, $|H_1(j\omega)|$ is 1 and independent of ω. This is an all-pass system discussed in section 9.4.3. Therefore,

$$|H_1(j\omega)| = 1.$$

This can be, of course, clearly seen from the equation as well:

$$\begin{aligned} |H_1(j\omega)| &= \left|\frac{j\omega-b}{j\omega+b}\right| = \frac{|j\omega-b|}{|j\omega+b|} \\ &= \frac{\sqrt{\omega^2+b^2}}{\sqrt{\omega^2+b^2}} \\ \therefore |H_1(j\omega)| &= 1. \end{aligned}$$

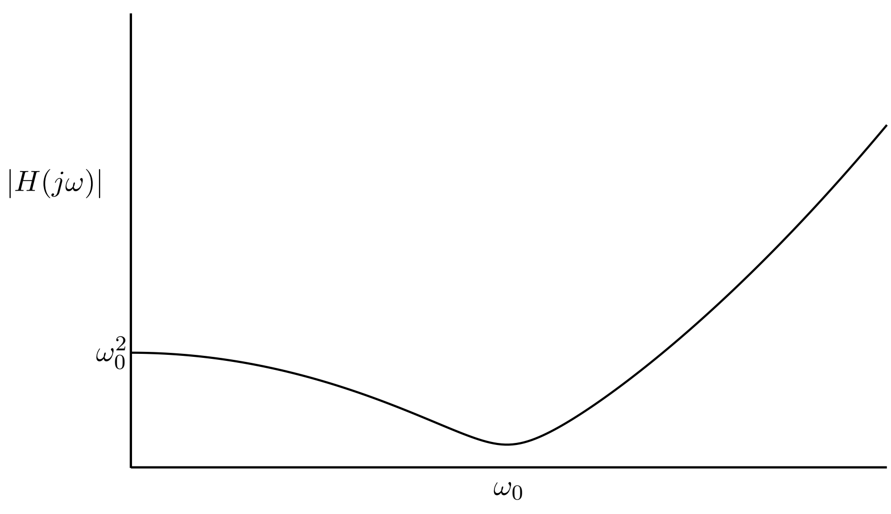
The pole-zero diagram of $H_2(s)$ is shown below. It has 2 zeros in the left half plane, and as we assume at the beginning $\omega_0 > a$.



Thus, we can see $H_2(s)$ as an inverse of a second order system with small damping ratio:

$$H_2(s) = s^2 + 2as + a^2 + \omega_0^2.$$

While $\omega << \omega_0$, $H_2(j\omega)$ remains roughly equal to ω_0^2. As ω approaches ω_0, $|H_2(j\omega)|$ attains its minimum since the length of the vector, $\overrightarrow{z}_1$ stemming from $s = -a + j\omega_0$ becomes shortest. However, because of the effect of the other zero, $|H_2(j\omega)|$ does not achieve its minimum exactly at $\omega = \omega_0$, but at ω slightly smaller than ω_0. In the vicinity of $\omega = \omega_0$, the length of the vector from the other zero, $\overrightarrow{z}_2$ does not change much. Thus, the variation of $|H_2(j\omega)|$ depends mostly on how the $\overrightarrow{z}_1$ changes its length. For $\omega >> \omega_0$, $|H_2(j\omega)|$ increases quadratically against linear ω. Thus, the overall magnitude plot is identical to that of $H_2(j\omega)$ and is sketched below for $\omega > 0$:



Note that the scaling on ω and $|H(j\omega)|$ axes are not the same.

Home study 2 O&W 9.40

Because we are dealing with a system which has initial conditions, it is easier to use the unilateral Laplace transform. From the properties of the unilateral Laplace transform, we get the following relationships:

$$\begin{aligned}
y(t) &\longleftrightarrow \mathcal{Y}(s) \\
y'(t) &\longleftrightarrow s\mathcal{Y}(s) - y(0^-) \\
y''(t) &\longleftrightarrow s^2\mathcal{Y}(s) - sy(0^-) - y'(0^-) \\
y'''(t) &\longleftrightarrow s^3\mathcal{Y}(s) - s^2y(0^-) - sy'(0^-) - y''(0^-)
\end{aligned}$$

Substituting these into the differential equation and solving for $\mathcal{Y}(s)$, we get

$$\mathcal{Y}(s) = \underbrace{\frac{\mathcal{X}(s)}{s^3+6s^2+11s+6}}_{ZSR} + \underbrace{\frac{s^2y(0^-) + s\left[y'(0^-)+6y(0^-)\right] + \left[y''(0^-)+6y'(0^-)+11y(0^-)\right]}{s^3+6s^2+11s+6}}_{ZIR}.$$

We have indicated in the previous equation that $\mathcal{Y}(s)$ can be split into two parts. The first part, the ZSR, corresponds to the output when the system is initially at rest and solely responds to the input. The second part, the ZIR, corresponds to the output when there is no input and the system responds only to its initial state.

(a) To determine the ZSR, we can simply use the equation derived on the previous page,

$$\begin{aligned}
\mathcal{Y}_{ZSR}(s) &= \frac{1}{(s^3+6s^2+11s+6)(s+4)} \\
&= \frac{1}{(s+1)(s+2)(s+3)(s+4)} = \frac{\frac{1}{6}}{s+1} - \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s+3} - \frac{\frac{1}{6}}{s+4}.
\end{aligned}$$

Taking the inverse transform gives

$$y_{ZSR}(t) = \frac{1}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t) - \frac{1}{6}e^{-4t}u(t).$$

(b) Using the ZIR portion derived previously, we can find the ZIR by substituting in the given i

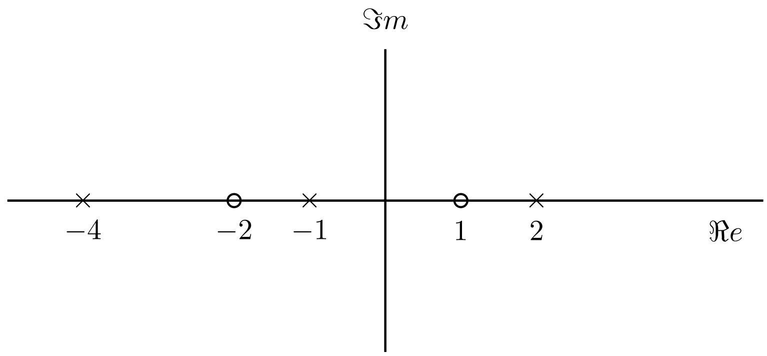
$$\mathcal{Y}_{ZIR}(s) = \frac{s^2+5s+6}{s^3+6s^2+11s+6} = \frac{1}{s+1}.$$

$$y_{ZIR}(t) = e^{-t}u(t)$$

(c) Then, by superposition,

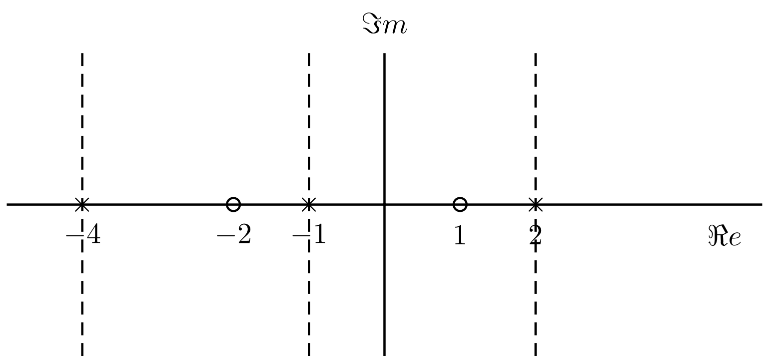
$$\begin{aligned}
y(t) &= y_{ZSR}(t) + y_{ZIR}(t) \\
&= \frac{7}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t) - \frac{1}{6}e^{-4t}u(t).
\end{aligned}$$

Problem 1 Consider an LTI system for which the system function $H(s)$ is rational and has the pole-zero pattern shown below:



(a) Indicate all possible ROC's that can be associated with this pole-zero pattern.

The ROCs are bounded by vertical lines through the location of the poles as in the figure below:



Thus, the possible ROCs are:

$$
\begin{aligned}
& \Re e\{s\} && < -4 \\
-4 < \; & \Re e\{s\} && < -1 \\
-1 < \; & \Re e\{s\} && < 2 \\
2 < \; & \Re e\{s\} &&
\end{aligned}
$$

(b) For each ROC identified in Part (a), specify whether the associated system is stable and/or causal.

Causal systems have ROCs that are to the right of the right-most pole. Stable systems are systems whose ROCs include the $j\omega$-axis.

$\Re e\{s\} < -4$: Not Causal. Not Stable

$-4 < \Re e\{s\} < -1$: Not Causal. Not Stable

$-1 < \Re e\{s\} < 2$: Not Causal. Stable

$2 < \Re e\{s\}$: Causal. Not Stable

Problem 2 Draw a direct-form representation for the causal LTI system with system function

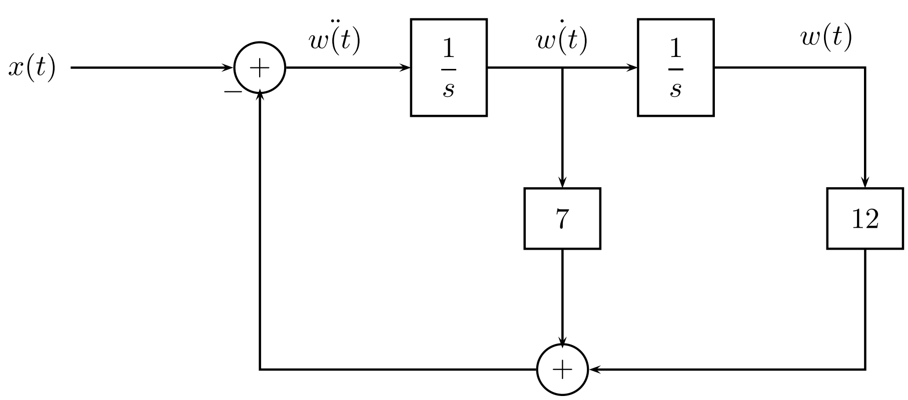
$$H(s) = \frac{s(s+1)}{(s+3)(s+4)}.$$

Note that the system function can be represented as follows:

$$\begin{aligned} H(s) &= \frac{s(s+1)}{(s+3)(s+4)} \\ &= \frac{s^2+s}{s^2+7s+12} \\ &= \frac{Y(s)}{W(s)}\frac{W(s)}{X(s)} \\ &= \underbrace{s^2+s}_{\frac{Y(s)}{W(s)}}\underbrace{\frac{1}{s^2+7s+12}}_{\frac{W(s)}{X(s)}}. \end{aligned}$$

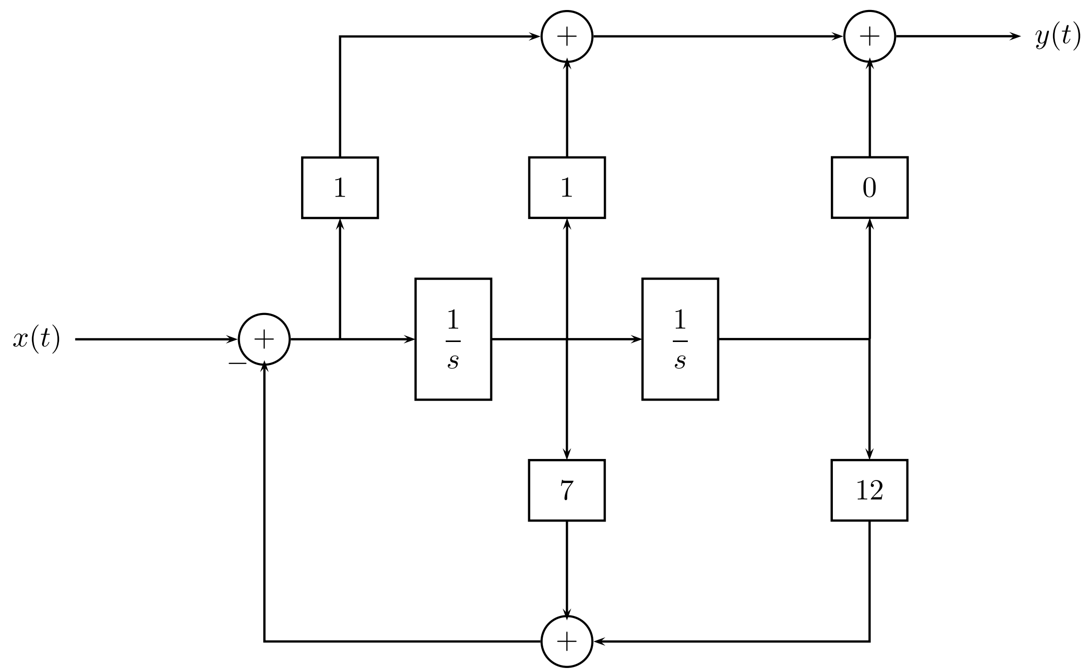
Thus, we can see the system $H(s)$ as a cascade of two systems, i.e., $Z(s) = \frac{Y(s)}{W(s)}$ which accounts for the zeros and $P(s) = \frac{W(s)}{X(s)}$ which accounts for the poles.

First, let's draw a block diagram representation of the system $P(s)$. Since the system is of second order, we would like to represent the system using only two integrators in cascade.



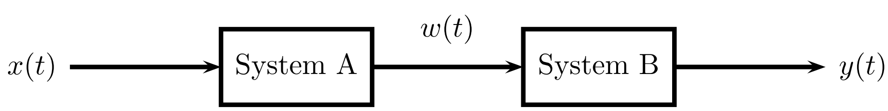
Here, note that $\ddot{w}(t) = \frac{d^2w(t)}{dt^2}$ and $\dot{w}(t) = \frac{dw(t)}{dt}$. It is easy to confirm that the above implementation of the system indeed represents the system $P(s)$ by recursively applying Black's formula.

Now, we would like to use the system above to implement the other system, $Z(s)$. Note that s in Laplace domain corresponds to differentiation in time domain. Thus, $y(t)$ is nothing but a linear combination of different orders of derivatives of $w(t)$; in our case, $y(t) = 1\times\ddot{w}(t)+1\times\dot{w}(t)+0\times w(t)$. Thus, a direct form representation of the overall system $H(s)$ is as shown below:



From this representation of the system, the numbers in the gain boxes are picked off simply from the coefficients of the numerator and denominator for a given rational system function. Often, when the gain terms are 1, they are omitted. When the gain terms are 0, the branch is often omitted.

Problem 3 Consider the cascade of two LTI systems as depicted below:



where we have the following:

- System A is causal with impulse response

$$h(t) = e^{-2t}u(t)$$

- System B is causal and is characterized by the following differential equation relating its input, $w(t)$, and output, $y(t)$:

$$\frac{dy(t)}{dt} + y(t) = \frac{dw(t)}{dt} + \alpha w(t)$$

- If the input $x(t) = e^{-3t}$, the output $y(t) = 0$.

1. Find the system function $H(s) = Y(s)/X(s)$, determine its ROC and sketch its pole-zero pattern. Note: Your answer should only have numbers in them (i.e., you have enough information to determine the value of α).

2. Determine the differential equation relating $y(t)$ and $x(t)$.

Solutions:

1. The overall system function $H(s)$ is $H_A(s) \times H_B(s)$ since the systems' A and B are cascaded together. From O&W Table 9.2 Laplace transforms of elementary functions,

$$H_A(s) = \frac{1}{s+2}, \Re e\{s\} > -2.$$

$H_B(s)$ is determined by letting $x(t) = e^{st}$ in the differential equation given for system B. Then $y(t) = H(s)e^{st}$ and we find

$$H_B(s) = \frac{s+\alpha}{s+1}, \Re e\{s\} > -1.$$

The ROC was known to be $\Re e\{s\} > -1$ and not $\Re e\{s\} < -1$ because we are told system B is causal. We need to solve for α. Along with the differential equation relating the input to the output for system B, we are told that if $x(t) = e^{-3t}$, an eigenfunction of an LTI system, then $y(t) = 0$.

Here, the concept we wished to test was the eigenfunction property of LTI systems. However, this problem was ill-posed. Namely, in order to apply the eigenfunction property, -3 from e^{-3t}, should have been in the ROC of $H(s)$ so that $H(-3)$ had existed. However, since the ROC of $H(s)$ is $\Re e\{s\} > -1$, $H(-3)$ does not exist. Therefore, you did not have enough information to determine $H(s)$ for this problem. The solution below is just to illustrate a possible method to obtain $H(s)$ assuming that $H(-3)$ existed.

Thus, $H_B(s)|_{s=-3} = 0$. This constraint allows us to solve for α. Specifically,

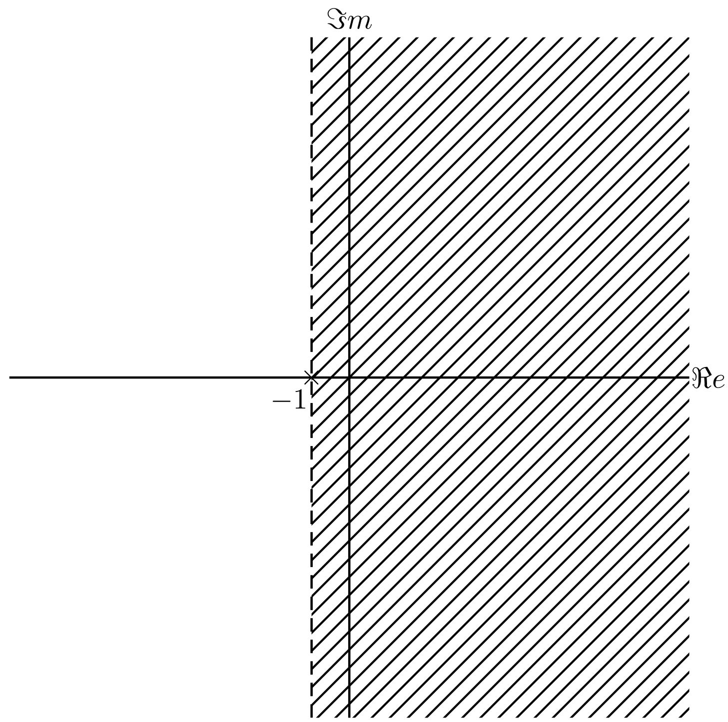
$$H_B(-3) = \frac{-3+\alpha}{-3+1} = 0 \longrightarrow \alpha = 3.$$

Therefore, the overall cascaded system function is

$$H(s) = \frac{s+3}{(s+1)(s+2)}, \Re e\{s\} > -1$$

The ROC must not have any poles in it and so the overall ROC must be to the right of the all poles in the system.

The pole-zero plot is shown below.



2. We can determine the differential equation relating $y(t)$ and $x(t)$ from the system function found in (a). Since $H(s) = \frac{Y(s)}{X(s)}$, we multiply the denominator of $H(s)$ by $Y(s)$ and we multiply the numerator of $H(s)$ by $X(s)$:

$$Y(s)(s^2 + 3s + 2) = X(s)(s + 3).$$

Distributing on both sides:

$$s^2Y(s) + 3sY(s) + 2Y(s) = sX(s) + 3X(s)$$

Because of linearity, we can take the inverse Laplace transform of each term above and get the differential equation:

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} + 3x(t).$$

Problem 4 Suppose we are given the following information about a causal and stable LTI system with impulse response $h(t)$ and a rational function $H(s)$:

- The steady state response to a unit step, i.e., $s(\infty) = \frac{1}{3}$.
- When the input is $e^t u(t)$, the output is absolutely integrable.

- The signal

$$\frac{d^2h(t)}{dt^2} + 5\frac{dh(t)}{dt} + 6h(t)$$

  is of finite duration.

- $h(t)$ has exactly one zero at infinity.

Determine $H(s)$ and its ROC.

Solution: To determine $H(s)$ and its ROC, we need to analyze and combine all the information given.

The first piece of information we are given is that the system is causal and stable. Because the system is causal, we know that the ROC is right-sided. Because the system is stable, we know that the ROC includes the $j\omega$-axis.

The next piece of information, *The steady state response to a unit step, i.e.,* $s(\infty) = \frac{1}{3}$, gives us information about $H(s)|_{s=0}$. Note that the step response is $s(t) = \int_{-\infty}^{t} h(\tau)\mathrm{d}\tau$. Thus, for $t = \infty$, $s(\infty) = \int_{-\infty}^{\infty} h(\tau)\mathrm{d}\tau$. But this is the same Laplace transform equation that can be used to solve for $H(s)|_{s=0}$. Thus, $H(0) = \frac{1}{3}$.

The next piece of information, *When the input is* $\mathrm{e}^t u(t)$*, the output is absolutely integrable,* gives us information about a zero of $H(s)$. We know that if $x(t) = \mathrm{e}^t u(t)$ then $X(s) = \frac{1}{s-1}, \Re e\{s\} > 1$ and $Y(s) = H(s)X(s)$. The ROC for $Y(s)$ will be *at least* the intersection of the ROC for $X(s)$ with the ROC for $H(s)$. If $y(t)$ is absolutely integrable, then we can take a Fourier transform of it, i.e., the ROC includes the $j\omega$-axis. Because of Property 2 in Chapter 9 of O&W, the ROC of any system, $Y(s)$ included, does not include any poles. Thus, the pole in $Y(s)$ at $s = 1$ is eliminated by having a zero at $s = 1$ in $H(s)$.

Jumping to the fourth bullet point, *h(t) has exactly one zero at infinity,* gives us information about the relative orders of the numerator and denominator for a rational tranform. Specifically, the order of the denominator is one greater than the order of the numerator. Thus, we know that the denominator has two poles.

The third bullet point gives us information about the poles of $H(s)$. By Property 3 in Chapter 9 of O&W, if a signal is of finite duration and is absolutely integrable then the ROC of the signal is the entire s-plane. We know that $\dfrac{d^2h(t)}{dt^2} + 5\dfrac{dh(t)}{dt} + 6h(t)$ is of finite duration. Is it also absolutely integrable? We can show that it is by looking individually at each of the terms in the function. We know that $h(t)$ is absolutely integrable because it is stable, i.e., its ROC includes the $j\omega$-axis. Multiplying $h(t)$ by 6 (a constant) will not change its absolute integrability. From Table 9.1, the ROC of the deriviative of a function includes the ROC of the original function. Thus, $5\frac{\mathrm{d}h(t)}{\mathrm{d}t}$ will include the $j\omega$-axis and be absolutely integrable. Likewise the second derivative, $\frac{\mathrm{d}^2h(t)}{dt^2}$ will include the ROC of the first derivative and thus, it is absolutely integrable also. From Table 9.2, the sum of 3 functions has at least the intersection of the ROC's of each of the three functions. Since the ROC of each of these functions includes the $j\omega$-axis, the sum will include the $j\omega$-axis and thus, the sum is absolutely integrable. Because the ROC of this signal, $\frac{d^2h(t)}{dt^2} + 5\frac{dh(t)}{dt} + 6h(t)$ is the entire s-plane, there can be no poles except at ∞. However, we know that $H(s)$ has at least two poles. In order for the signal to have no poles, we must make sure that the poles of $H(s)$ are cancelled by zeros of the signal. Taking the Laplace transform of the signal gives us

$$s^2H(s) + 5sH(s) + 6H(s) = (s+2)(s+3)H(s).$$

Thus, $H(s)$ can only have two poles, at $s = -2$ and $s = -3$.
Combining all the information about the poles and zeros gives us,

$$H(s) = K\frac{s-1}{(s+2)(s+3)}, \Re e\{s\} > -2.$$

Finally, we choose K such that $H(0) = \frac{1}{3}$. That is, $K = -2$ and

$$H(s) = -2\frac{s-1}{(s+2)(s+3)}, \Re e\{s\} > -2.$$

Problem 5 Consider the basic feedback system of Figure 11.3 (a) on p.819 of O&W. Determine the closed-loop system impulse response when

$$H(s) = \frac{1}{s+5}, \quad G(s) = \frac{2}{s+2}.$$

We can use Black's Formula to compute the system function for the entire system, $Q(s)$. This is given by:

$$\begin{aligned} Q(s) &= \frac{\frac{1}{s+5}}{1 + \frac{2}{(s+5)(s+2)}} \\ &= \frac{(s+2)}{s^2 + 7s + 12} \end{aligned}$$

Next, we can use partial fraction expansion to write $Q(s)$ as a sum of 1st order terms.

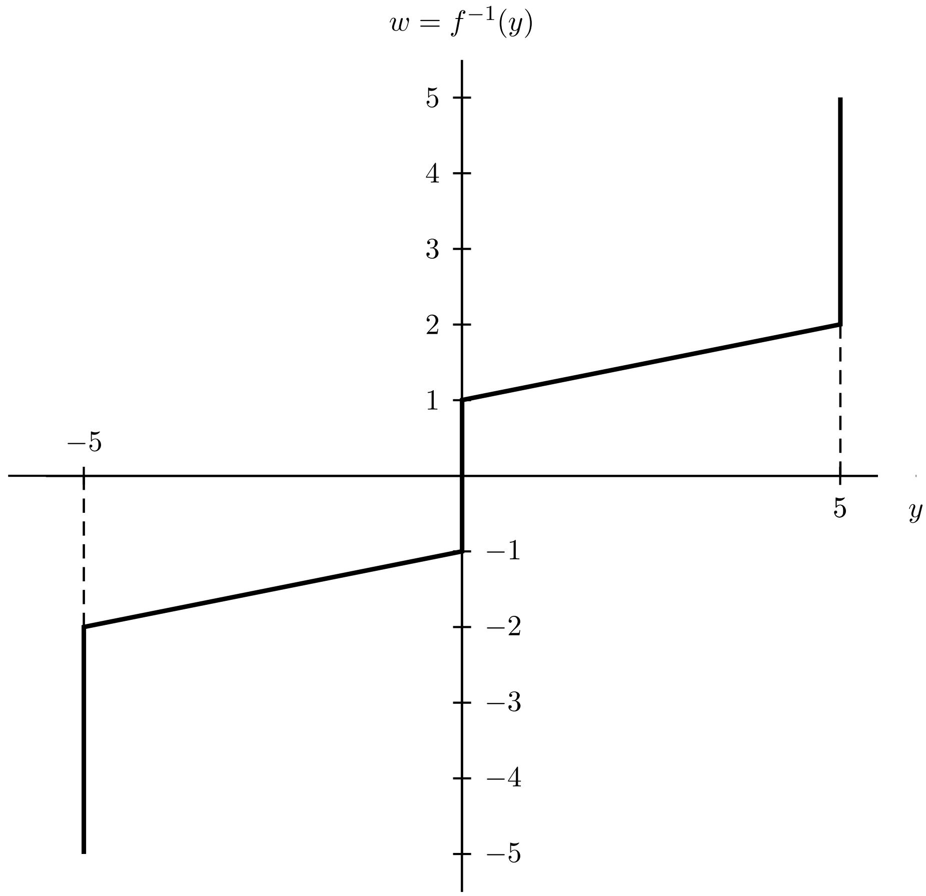
$$\begin{aligned} Q(s) &= \frac{s+2}{s^2 + 7s + 12} \\ &= \frac{s+2}{(s+4)(s+3)} \\ &= \frac{2}{s+4} - \frac{1}{s+3} \end{aligned}$$

Since this is a feedback system, we know it is causal. Thus, we find the inverse Laplace transform for the system function using the causal part to obtain the inpulse response as follows:

$$q(t) = 2e^{-4t}u(t) - e^{-3t}u(t)$$

Problem 6

(a) $f^{-1}(y)$ is shown below.



(b) From the feedback system diagram, we can write

$$\begin{aligned} w(t) &= K_1(x(t) - K_2 y(t)) \\ &= K_1 x(t) - K_1 K_2 y(t) \\ f^{-1}(y) &= K_1 x(t) - K_1 K_2 y(t) \end{aligned}$$

Therefore,

$$x(t) = \frac{1}{K_1} f^{-1}(y) + K_2 y(t)$$

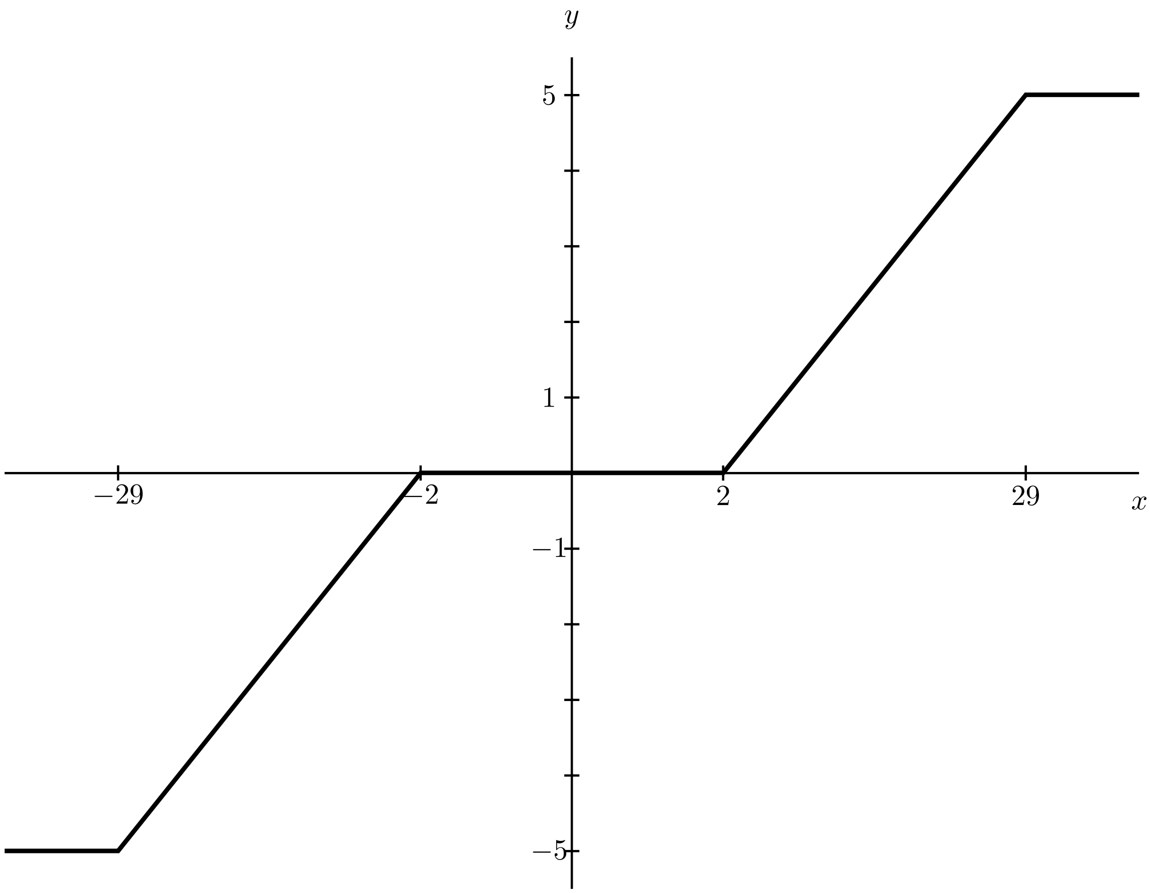
(c) 1. $K_1 = 0.5$ and $K_2 = 5$. We can write the input-output relation found in part b,

$$x(t) = 2f^{-1}(y) + 5y(t)$$

Let us evaluate the above function for some values of y

y	$f^{-1}(y)$	x
0	-1 to -1	-2 to 2
5	2.0	29
-5	-2.0	-29

Note that y saturates at $y = -5$ and $y = 5$. Following is the plot of y as a function of x.



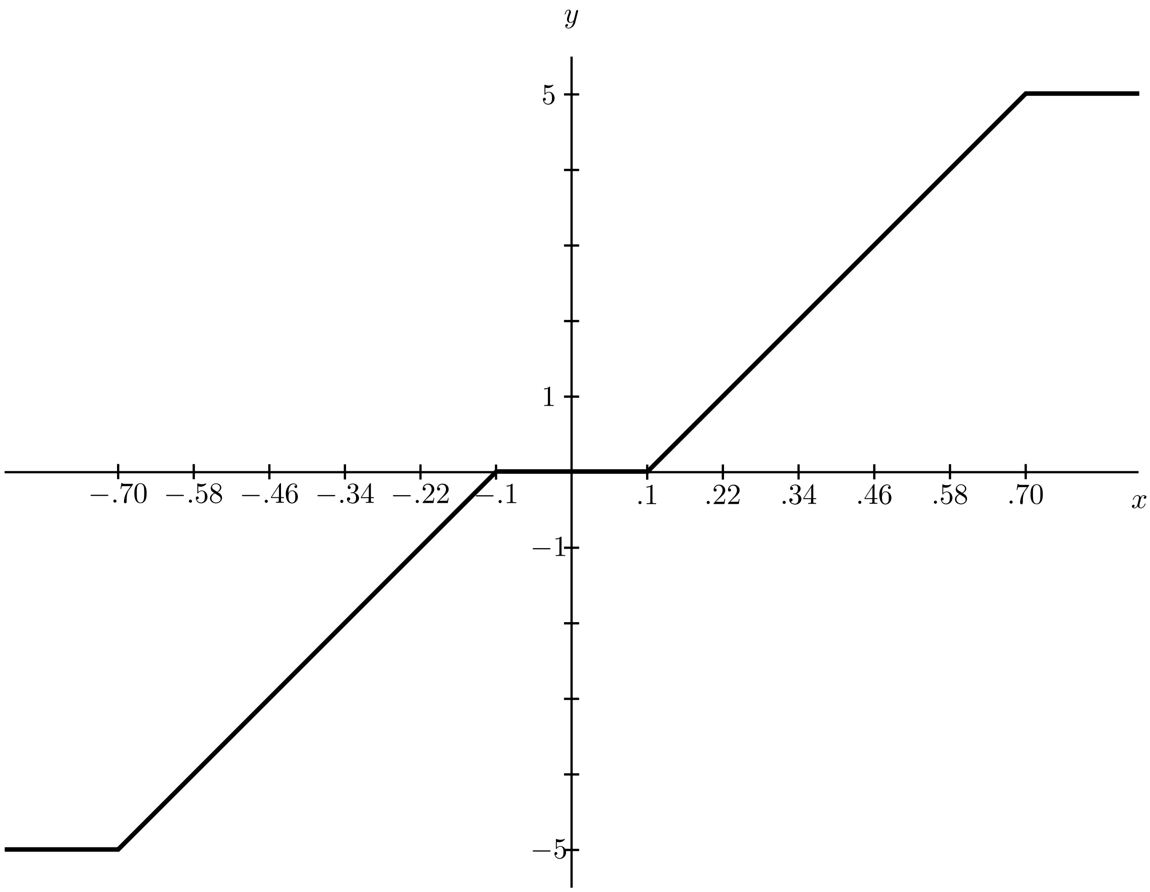
2. Now $K_1 = 10$ and $K_2 = 0.1$. We can write the input-output relation found in part b,

$$x(t) = \frac{1}{10} f^{-1}(y) + 0.1 y(t)$$

Let us evaluate the above function for some values of y

y	$f^{-1}(y)$	x
0	-1 to -1	$-\frac{1}{10}$ to $\frac{1}{10}$
1	1.2	0.22
2	1.4	0.34
3	1.6	0.46
4	1.8	0.58
5	2.0	0.70

For negative values of y, we would get negative values of x. Note that y saturates at $y = -5$ and $y = 5$. Following is the plot of y as a function of x.



(d) We see from the relation derived in part (b), that if K_1 is large, we can find the approximate linear relation between input, $x(t)$, and output, $y(t)$.

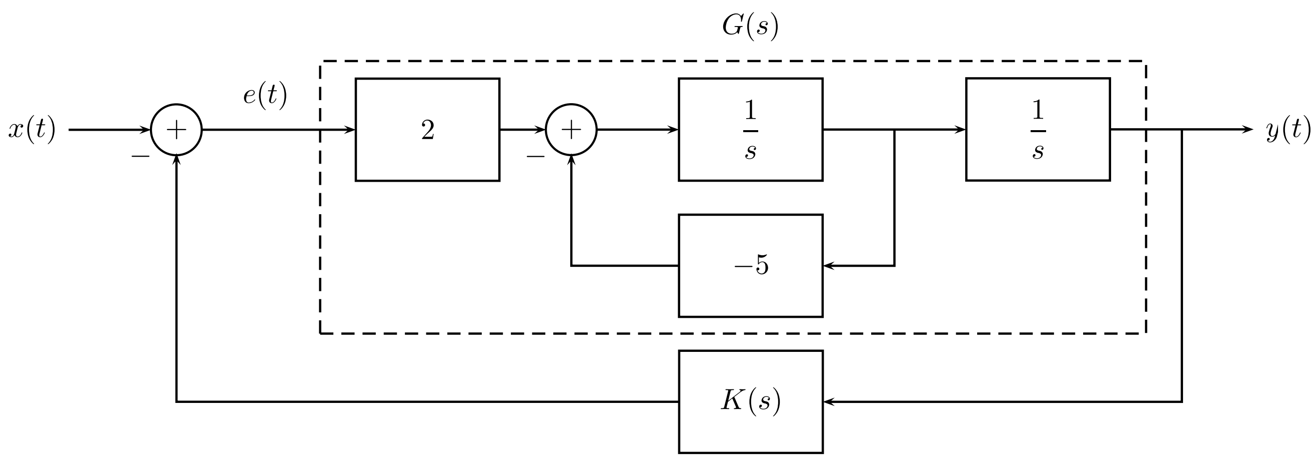
$$y(t) \cong \frac{1}{K_2} x(t)$$

Therefore, K_1 needs to be large in magnitude to have approximated linear relation between the input and the output.

From part (c), we see that the slope in linear range approximately depends on the magnitude of K_2. Therefore, K_2 is a design parameter whose value is set by the user. However, as K_2 decreases, saturation limits are hit more easily, so for practical design point of view, we may not want to decrease K_2 too much.

Problem 7

1. We need to find $G(s) = \frac{Y(s)}{E(s)}$. $G(s)$ is as shown below in the overall system diagram,



Using the closed-loop system function formula (equation 11.1 in the textbook) for the feedback system, assume $F(s)$, inside $G(s)$, we find

$$F(s) = \frac{\frac{1}{s}}{1 + \frac{1}{s}(-5)} = \frac{1}{s-5}$$

Therefore,

$$G(s) = \frac{Y(s)}{E(s)} = 2 \times F(s) \times \frac{1}{s}$$

$$G(s) = \frac{2}{s(s-5)}$$

The system function $G(s)$ has a pole in the right half of s-plane ($s = 5$). Therefore, the system is not stable.

2. We are given $K(s) = K_p$ where K_p is a real constant. Using the Black's furmula (equation

11.1 from the textbook) for the feedback system, we find

$$\begin{aligned}
H(s) &= \frac{G(s)}{1+K(s)G(s)} \\
&= \frac{G(s)}{1+K_pG(s)} \\
&= \frac{\frac{2}{s(s-5)}}{1+K_p\frac{2}{s(s-5)}} \\
&= \frac{2}{s(s-5)+2K_p} \\
&= \frac{2}{s^2-5s+2K_p}
\end{aligned}$$

The roots of the denominator are the poles of $H(s)$ that need to be in the left half of the s-plane for $H(s)$ to be stable.

$$\begin{aligned}
s^2-5s+2K_p &= 0 \\
s = \frac{5 \pm \sqrt{25-8K_p}}{2} &= \frac{5}{2} \pm \frac{\sqrt{25-8K_p}}{2}
\end{aligned}$$

If $\sqrt{25-8K_p}$ is real, there are 2 poles. If $\sqrt{25-8K_p} < 5$, the two poles are in the right half of s-plane. If $\sqrt{25-8K_p} = 5$, both poles are at $s = 0$. If $\sqrt{25-8K_p} > 5$, one pole is in the right half plane and another in the left half.
If $\sqrt{25-8K_p}$ is imaginary, the poles are complex conjugate of each other and are in the right half of the s-plane.
Therefore, the system cannot be stabilized by changing K_p.

3. We are given $K(s) = K_ds + K_p$ where both K_d and K_p are real constants. Now, we have,

$$\begin{aligned}
H(s) &= \frac{G(s)}{1+K(s)G(s)} \\
&= \frac{G(s)}{1+(K_ds+K_p)G(s)} \\
&= \frac{\frac{2}{s(s-5)}}{1+(K_ds+K_p)\frac{2}{s(s-5)}} \\
&= \frac{2}{s(s-5)+2(K_ds+K_p)} \\
&= \frac{2}{s^2-(5-2K_d)s+2K_p}
\end{aligned}$$

The roots of the denominator are the poles of $H(s)$ that need to be in the left half of the s-plane for $H(s)$ to be stable.

$$\begin{aligned}
s^2-(5-2K_d)s+2K_p &= 0 \\
s = \frac{(5-2K_d) \pm \sqrt{(5-2K_d)^2-8K_p}}{2} &= \frac{5-2K_d}{2} \pm \frac{\sqrt{(5-2K_d)^2-8K_p}}{2}
\end{aligned}$$

To find the range of K_d and K_p that will stabilize the system, we use the Routh-Hurwitz criteria for left half plane roots on the denominator polynomial for which we have following requirements:

$$-(5 - 2K_d) > 0$$
$$K_d > \frac{5}{2}$$

We also need,

$$2K_p > 0$$
$$K_p > 0$$