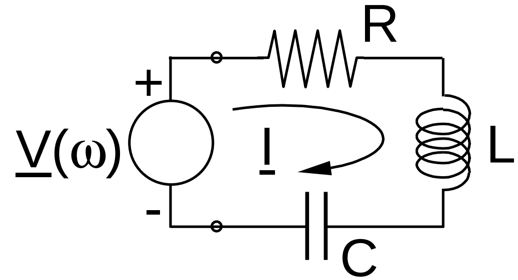
RESONATORS

Impedance [$v(t) = R_e\{\underline{V}e^{j\omega t}\}$]:

$v = RI$	$\rightarrow$	$\underline{V} = R\underline{I}$	$\rightarrow$	$\underline{Z} = R$
$v = L\, di/di$	$\rightarrow$	$\underline{V} = j\omega L\underline{I}$	$\rightarrow$	$\underline{Z} = j\omega L$
$i = C\, dv/dt$	$\rightarrow$	$\underline{I} = j\omega C\underline{V}$	$\rightarrow$	$\underline{Z} = 1/j\omega C$

Series RLC Resonance:

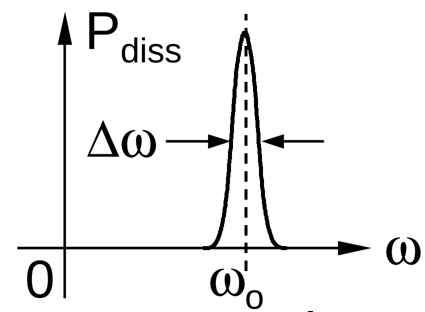


$$\underline{V}(\omega) = \underline{I}(\omega)[R + j\omega L + (j\omega C)^{-1}]$$
$$= \text{min if } j\omega L = -1/j\omega C$$

$$\omega_o = (LC)^{-0.5}$$

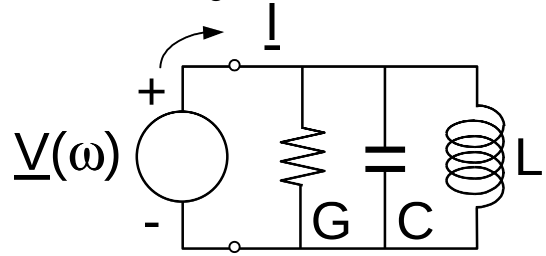
$$Q = \omega_o w_T/P_{diss} \cong \omega_o/\Delta\omega \cong \omega_o L/R$$

where:



$$w_T = L|\underline{I}|^2/4 + C|\underline{V}|^2/4 = L|\underline{I}|^2/2\,[J] \text{ at } \omega_o$$
$$P_{diss} = |\underline{I}|^2 R/2\ [W]$$

Parallel RLC Resonance:

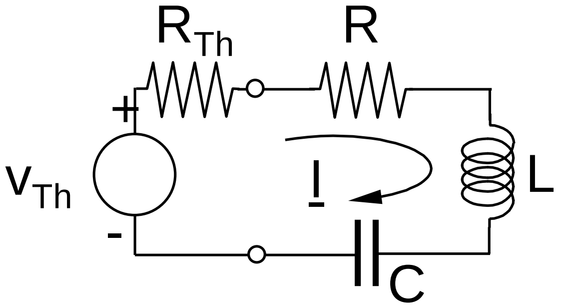


$$\underline{I}(\omega) = \underline{V}(\omega)[G + j\omega C + (j\omega L)^{-1}]$$
$$= \text{min if } j\omega C = -1/j\omega L, \text{ i.e. if}$$

$$Q = \omega_o C/G$$
$$\omega_o = (LC)^{-0.5}$$

COUPLING TO RESONATORS

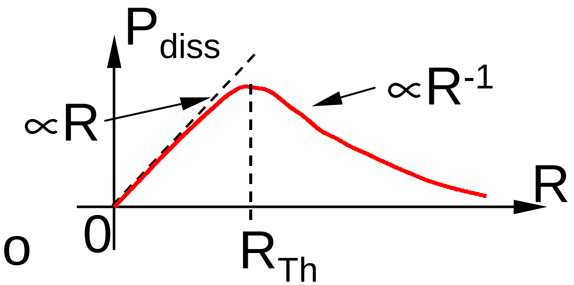
RLC Resonators:



To maximize power dissipated in R:
Set $R = R_{Th}$ at $\omega = \omega_o = (LC)^{-0.5}$

Proof that $R = R_{Th}$ for Maximum Power Dissipation:

$P_{diss} = i^2R = [v_{Th}/(R + R_{Th})]^2R$
Set $\partial P_{diss}/\partial R = 0$ to maximize P_{diss}



$\partial P_{diss}/\partial R = v_{Th}^2[(R + R_{Th})^{-2} - 2R(R + R_{Th})^{-3}] = 0$, so

$R + R_{Th} = 2R \Rightarrow$ **$R = R_{Th}$ for maximum P_{diss}** "matched load"

Conversely, resonator loses power fastest if $R_L = R(v_{Th} = 0)$

Resonator Internal Q_I, External Q_E, and Loaded Q_L:

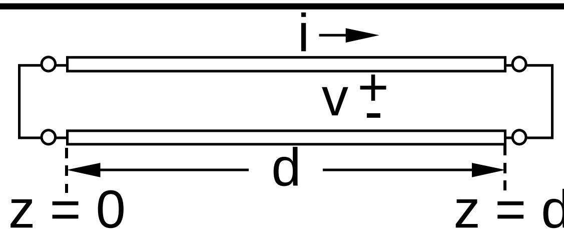
$Q = \omega_o w_T P_{diss} \Rightarrow Q_I = \omega_o L/P_{dissI}$, $Q_E = \omega_o L/P_{dissE}$, $Q_L = \omega_o L/P_{dissL}$

Therefore: $Q_L^{-1} = Q_I^{-1} + Q_E^{-1}$ and **$Q_I = Q_E$ for maximum P_{diss}**

TEM RESONATORS

Voltages and Currents:

e.g.:



$v(t,z) = V_+ \cos(\omega t - kz) + V_- \cos(\omega t + kz) = 0$ at $z = 0, d$

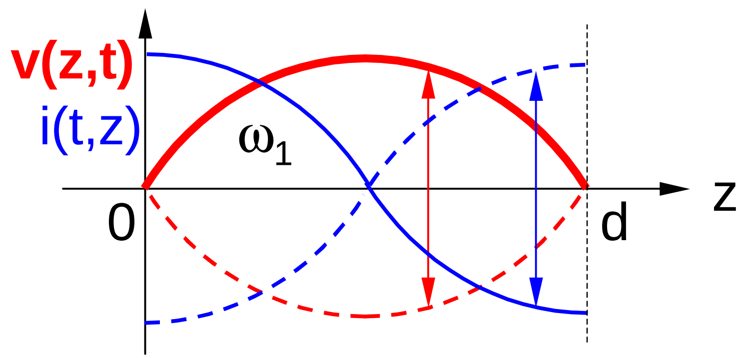
Boundary conditions $\Rightarrow V_- = -V_+$

$v(t,z) = V_+[\cos(\omega t - kz) - \cos(\omega t + kz)] \quad = 2V_+ \sin \omega t \sin kz$

where $\cos \alpha - \cos \beta = -2 \sin[(\alpha + \beta)/2] \sin [(\alpha - \beta)/2]$

$i(t,z) = (V_+/Z_o)[\cos(\omega t - kz) + \cos(\omega t + kz)] = 2(V_+/Z_o)\cos \omega t \cos kz$

Resonant Frequencies ω_n:

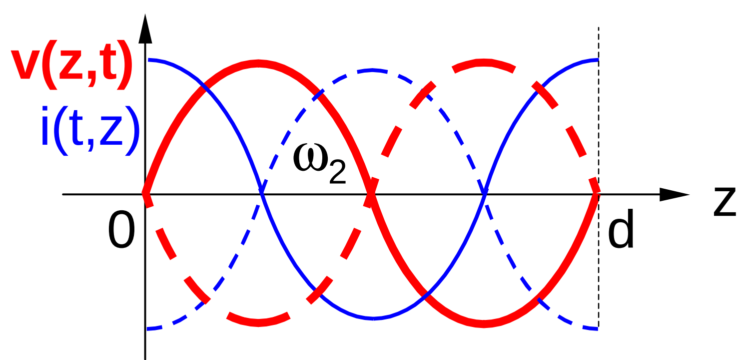


$k_n d = \omega_n d/c = n\pi, \; n = 0, 1, 2, \ldots$

$\omega_n = n\pi c/d$

$n\lambda_n/2 = d$

DC Resonance ($\omega = 0$):



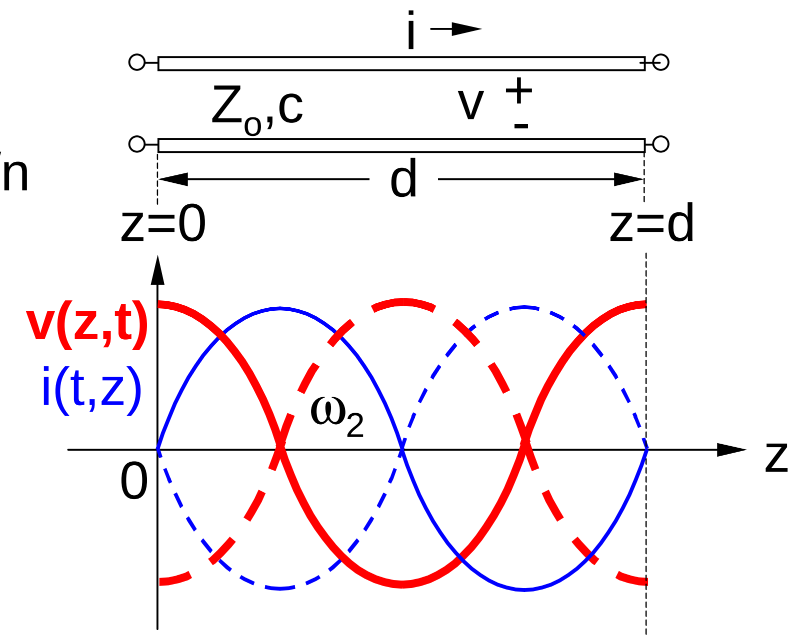
Current flows in loop,

Voltages are zero

$w_e = 0, w_m \neq 0$ [J]

MORE TEM RESONANCES

Open-Circuit Resonator:



$n\lambda_n/2 = d, \;\; n = 0,1,2,\ldots \Rightarrow \lambda_n = 2d/n$

$\omega_n = n\pi c/d$

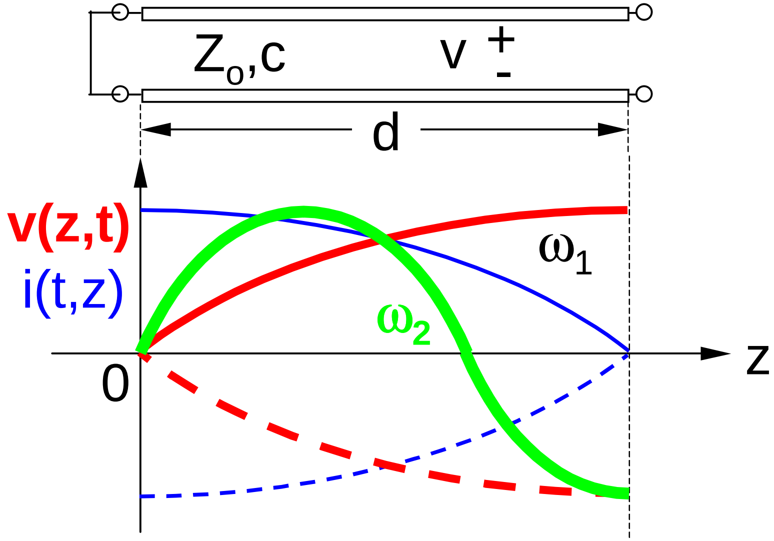
$\lambda_0 = \infty \quad \rightarrow \quad \omega_o = 0$

$\lambda_1 = 2d \quad \rightarrow \quad \omega_1 = \pi c/d$

$\lambda_2 = d \quad \rightarrow \quad \omega_2 = 2\pi c/d$

$\lambda_3 = 3d/2 \quad \rightarrow \quad \omega_3 = 3\pi c/d$

Quarter-Wave Resonator:



$n\lambda_n/2 - \lambda_n/4 = d = (n - 0.5)\lambda_n/2$

$\Rightarrow \lambda_n = 2d/(n - 0.5)$

$\lambda_1 = 4d$

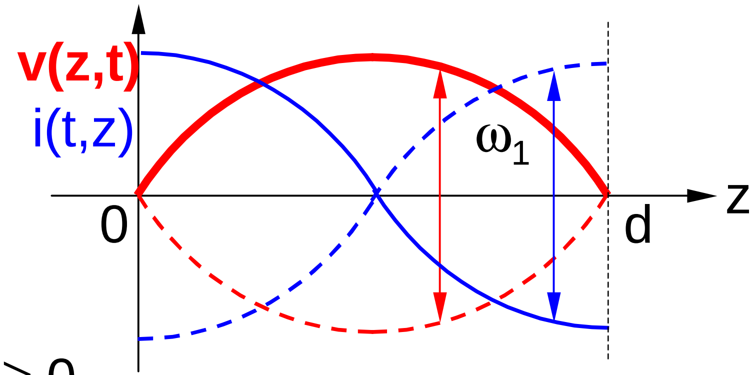
$\lambda_2 = 4d/3$

$\lambda_3 = 4d/5$

$\lambda_4 = 4d/7$

ENERGY IN TEM RESONATORS

Electric Energy Density W_e:



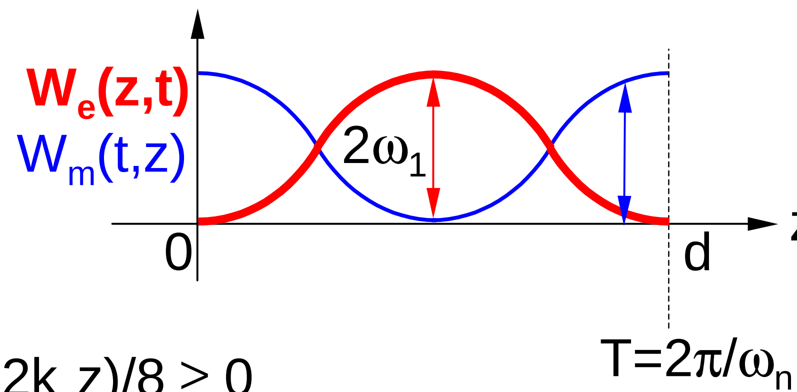
$v(t,z) = V_o \sin \omega_n t \sin k_n z$

$w_e(t,z) = Cv^2/2$

$= CV_o^2 (\sin^2\omega_n t)(\sin^2 k_n z)/2$

$= CV_o^2(1 - \cos 2\omega_n t)(1 - \cos 2k_n z)/8 \geq 0$

Magnetic Energy Density W_m:



$i(t,z) = (V_o/Z_o)\cos \omega_n t \cos k_n z$

$w_m(t,z) = Li^2/2$

$= L(V_o/Z_o)^2(\cos^2\omega_n t)(\cos^2 k_n z)/2$

$= L(V_o/Z_o)^2(1 + \cos 2\omega_n t)(1 + \cos 2k_n z)/8 \geq 0$

Electric and Magnetic Energies w_e and w_m:

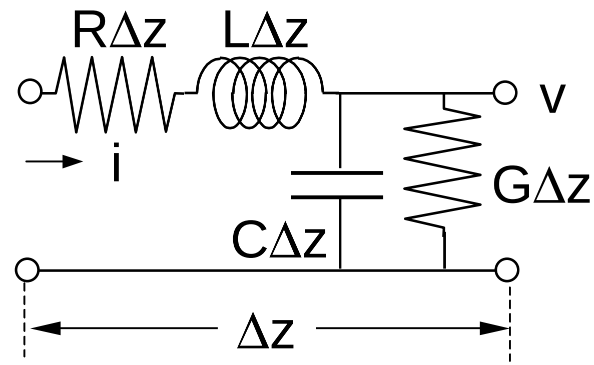
$w_e(t) = CV_o^2(1 - \cos 2\omega_n t)d/8, \quad \langle w_e \rangle = CV_o^2 d/8$

$w_m(t) = L(V_o/Z_o)^2(1 - \cos 2\omega_n t)d/8, \quad \langle w_m \rangle = L(V_o/Z_o)^2 d/8$

$\langle w_e \rangle = \langle w_m \rangle$ because $C = L/Z_o^2 \quad [Z_o = (L/C)^{0.5}]$

LOSSES IN TEM RESONATORS

Distributed Losses:



$$P_{diss}(t,z) = i^2R + v^2G \ \ [\text{watts}/\text{m}]$$

$$P_d = \int_0^d \langle P_{diss} \rangle dz$$

$$= \int_0^d (V_o/Z_o)^2 \langle \cos^2 \omega_n t \rangle (\cos^2 k_n z) R dz$$

$$+ \int_0^d V_o^{\ 2} \langle \sin^2 \omega t \rangle (\sin^2 k_n z) G dz \ \ \text{(for short-circuited TEM)}$$

$$= Rd(V_o/Z_o)^2/4 + GdV_o^{\ 2}/4 \quad [\text{watts}]$$

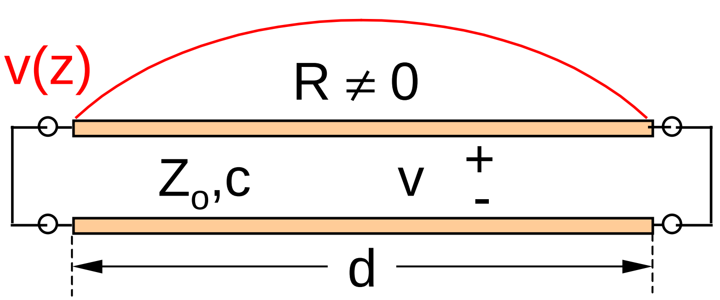
Example of Q Calculation (n =1):

$$Q_I = Q_L = \omega_o w_T/P_d = \pi c C Z_o^2/Rd = \pi(LC)^{-0.5}C(L/C)/Rd = \pi Z_o/Rd$$

$$\omega_o(n=1) = \pi c/d$$

$$w_T = 2\langle w_e \rangle = CV_o^{\ 2}d/4$$

$$P_d = Rd(V_o/Z_o)^2/4$$

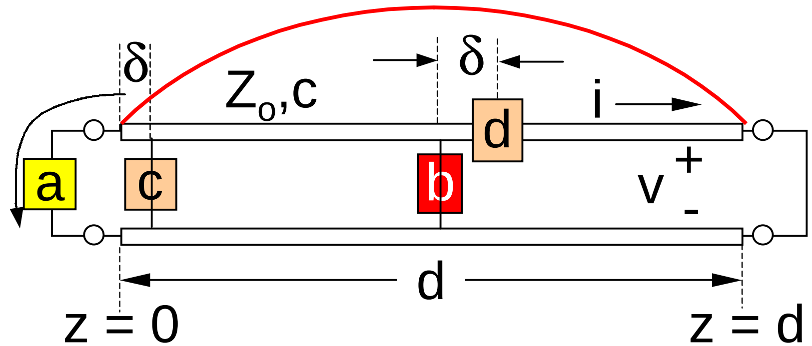


LUMPED LOSSES IN TEM RESONATORS

Lumped Losses a :

$i = V_o/Z_o \cos \omega t$

e.g. shorted TEM line, length d:



Perturbation technique: calculate losses using unperturbed v or i

Requires: $\Delta v << v_o$ or $\Delta i << i_o$; $\Gamma \cong \Gamma_o$

Examples: $G_{z=0} = (Z_n - 1)/(Z_n + 1) \cong G_{0(z=0)} = -1$, if $Z_{an} = R_a/Z_o << 1$

$G_{(z=d/2)} \cong G_o = 0$, if $Z_{bn} = R_b/Z_o >> 1$

$R_c >> |Z_{left}(z = \delta)| = |-jZ_o \sin k\delta|$ is perturbation

Note: $R_c = Z_o >> Z_o \sin k\delta$ if $2\pi\delta/\lambda << 1$

$R_d << |-jZ_o/\sin k\delta|$ is perturbation, so is $R_d = Z_o$ if $2\pi\delta << \lambda$

Loss Computation Examples:

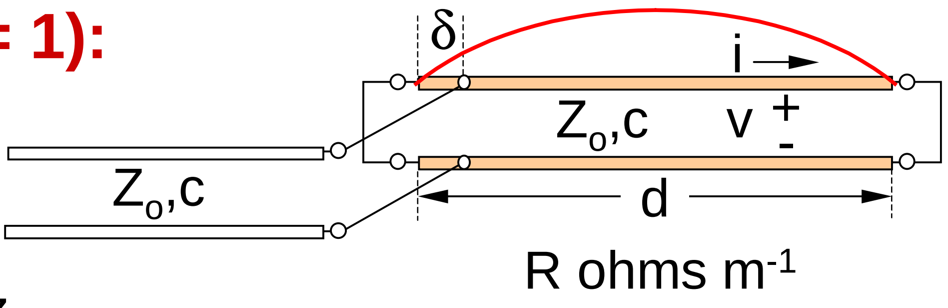
$P_d \cong \langle i^2{}_{z=0} \rangle R_a$ for resistor "a" (use unperturbed $i_{z=0}$)

$P_d \cong \langle v^2{}_{z=d/2} \rangle / R_b$ for resistor "b" (use unperturbed $v_{z=d/2}$)

$P_d \cong \langle (V_o \sin kd \, \sin \omega t)^2 \rangle / R_c$ for "c" (uses unperturbed v)

COUPLED TEM RESONATORS

Example, Q Computation (n = 1):



$$\omega_o(n=1) = \pi c/d$$

$$w_T = 2\langle w_e \rangle = CV_o^2 d/4$$

$$P_{dext} \cong \left\langle (V_o \sin k\delta \sin \omega t)^2 \right\rangle / Z_o$$

$$Q_I = \omega_o w_T / P_{dInternal} = \pi Z_o / Rd \quad \text{(see L18-6)}$$

$$Q_E = \omega_o w_T / P_{dExternal}$$

$$= (\pi c/d)\left(CV_o^2 d/4\right) Z_o \Big/ \left\langle (V_o \sin k\delta \sin \omega t)^2 \right\rangle$$

$$\cong \pi c C Z_o \lambda^2 / (2\pi\delta)^2 = \pi (LC)^{-0.5} C (L/C)^{0.5} (2d)^2 / (2\pi\delta)^2$$

$$= (d/\delta)^2 / \pi$$

Example, Coupling to TEM Resonator:

$Q_I = Q_E = 2Q_L$ for perfect coupling at resonance (matched load)

$\Rightarrow \pi Z_o/Rd \cong (d/\delta)^2/\pi \rightarrow$ Set $\delta \cong (d^3 R/Z_o \pi^2)^{0.5}$ [m]

$\Delta\omega = \omega_o/Q_L = \omega_o/2Q_I = (\pi c/d)/(2\pi Z_o/Rd) = cR/2Z_o$ [rs^{-1}]

Imperfect coupling if $Q_E \neq Q_I$; $Q_E/Q_I = R/R_{Th} \Rightarrow G$ at ω_o