6.013 Recitation 11

Quasistatic Electric and Magnetic Fields in Devices and Circuit Elements

A. Introduction

The behavior of most electric devices depends on static or slowly varying (quasistatic[1]) electric and magnetic fields, and so it is important to understand how these fields are determined and controlled. They generally can be computed in a straightforward manner when the charge density ρ and current density $\overline{J}$ are known everywhere. In other cases we may know or control the shapes of conductors and their electrostatic potentials Φ, but not the charges. Below are discussed general methods for finding these fields and for determining the resulting behavior of resistors and capacitors.

B. Determining Fields from Charges and Currents

We have seen earlier[2] how Faraday's law for a static situation, $\nabla \times \overline{E} = 0$, implies

$$\overline{E} = -\nabla\Phi \qquad (1)$$

where Φ is a any scalar potential field; this follows from the identity $\nabla\times(\nabla\Phi) = 0$. We also saw[2] that the potential Φ at position $\overline{r}$ was simply related to the charge distribution $\rho(\overline{r}')$ by:

$$\Phi(\overline{r}) = (1/4\pi\varepsilon_o) \int_{V'} [\rho(\overline{r}')/|\overline{r} - \overline{r}'|]\, dv' \qquad (2)$$

Thus a simple calculation yields the potential field $\Phi(\overline{r})$ if the charge distribution $\rho(\overline{r})$ is known, and then use of (1) yields $\overline{E}(\overline{r})$, the desired answer. The coordinates are suggested in Figure R13-1.

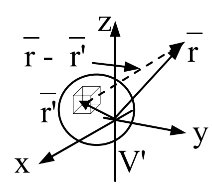

Figure R13-1. Coordinate system for finding $\Phi(\overline{r})$

Equation (2) states that the total electric potential Φ equals the sum of the contributions from each independent charge, so that superposition applies. We can also

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[1]The term quasistatic applies when the time variations are so slow that the corresponding wavelength $\lambda = c/f$ is very large compared to the device dimensions of interest. In this case the solutions are the same as the electrostatic or magnetostatic solutions, plus small perturbation fields due to the time variation. These perturbations are sometimes neglected and sometimes they are the principal effect of interest.

[2] In Lecture 3, and in Section 2.1 of the text, leading to Equations (2.1.21) and (2.1.27).

obtain a simple superposition equation relating $\rho(\bar{r}')$ and $\bar{E}(\bar{r})$ directly. We have seen[3] that the field contribution from a charge q at the origin is:

$$\bar{E}(\bar{r}) = \hat{r}\ q/4\pi\varepsilon_o r^2 \tag{3}$$

and so superposition yields the following useful relation:

$$\bar{E}(\bar{r}) = (1/4\pi\varepsilon_o) \int_{V'} [(\bar{r} - \bar{r}')\, \rho(\bar{r}')/|\bar{r} - \bar{r}'|^3]\, dv' \tag{4}$$

If we seek the magnetic field distribution produced by a given static current distribution $\bar{J}(\bar{r})$, we can take a similar approach. Since $\nabla \bullet \bar{B} = 0$, it follows from the identity $\nabla \bullet (\nabla \times \bar{A}) = 0$ that:

$$\bar{B} = \nabla \times \bar{A} \tag{5}$$

where we showed earlier[4] that $\bar{A}$ depends in a simple way on $\bar{J}$:

$$\bar{A}(\bar{r}) = (\mu_o/4\pi) \int_{V'} [\bar{J}(\bar{r}')/|\bar{r} - \bar{r}'|]\, dv' \tag{6}$$

Analogous to the electrostatic case (1-2), simple evaluation of (6) yields $\bar{A}$ if $\bar{J}(\bar{r}')$ is known, and then $\bar{B}$ can be readily found using (5).

Again superposition applies and the total field $\bar{B}$ is the sum of contributions from all current elements, so a simpler equation directly relating $\bar{J}(\bar{r}')$ to $\bar{B}(\bar{r})$ can be found, analogous to (4). Combining (5) and (6) yields:

$$\bar{H} = (1/4\pi) \int_{V'} \nabla \times \left[\bar{J}(\bar{r}')/|\bar{r} - \bar{r}'|\right] dv' \tag{7}$$

The curl in (7) operates on the product of a vector $\bar{J}$ and a scalar ψ related to $\bar{r}$, so we require the vector identity:

$$\nabla \times (\psi \bar{J}) = (\psi \nabla \times \bar{J}) + (\nabla\psi \times \bar{J}) \tag{8}$$

The first term on the right-hand side $(\psi\nabla \times \bar{J})$ equals zero because ∇ differentiates only with respect to $\bar{r}$, and $\bar{J}$ is a function only of $\bar{r}'$. To evaluate $\nabla\psi = \nabla|\bar{r} - \bar{r}'|^{-1}$ in the right-most term of (8) we need an expression for ∇ in spherical coordinates (r, θ, ϕ):

$$\nabla = \hat{r}\, \partial/\partial r + \hat{\theta}\, r^{-1} \partial/\partial\theta + \hat{\phi}\, (r \sin\theta)^{-1} \partial/\partial\phi \tag{9}$$

[3] In the text, (2.1.10) on page 48.
[4] Equation (2.1.27) on page 52 in the text.

Because ψ is a function only of r, only the first term of (9) is non-zero. Evaluating this term for $\nabla|\overline{r} - \overline{r}'|^{-1}$ becomes trivial if we temporarily let $\overline{r}' = 0$, which we can do because ∇ does not operate on $\overline{r}'$. Thus (8) becomes:

$$\nabla \times (\psi \overline{J}) = (\nabla\psi) \times \overline{J} = (\nabla | \overline{r} |^{-1}) \times \overline{J} = (-\hat{r}/r^2) \times \overline{J} \quad (10)$$

If we shift the origin of the coordinate system to $\overline{r}'$, then the last expression in (10) becomes $-\hat{R}/|\overline{r} - \overline{r}'|^2 \times \overline{J}(\overline{r}')$ where $\hat{R}$ is the unit vector in the direction of $\overline{r} - \overline{r}'$. It then follows from (7) that:

$$\overline{H} = (1/4\pi)\int_{V'}\left[\overline{J}(\overline{r}')\times\hat{R}\big/|\overline{r}-\overline{r}'|^2\right]dv' \quad (11)$$

This useful equation is called the *Biot-Savart law*.

Often, however, electrostatic potentials Φ on certain surfaces are given instead of the entire charge distribution $\rho(\overline{r})$, and other approaches to finding $\overline{E}(\overline{r})$ must be taken. Here we first present a simple visual sketch-based method for obtaining quick approximate solutions, and then present a more formal analytical approach.

The visual approach begins with recalling that the boundary conditions[5] on $\overline{E}$ at a perfect conductor require $\overline{E}_{//} = 0$. That is, $\overline{E}$ must be perpendicular to the surface, where $\hat{n} \bullet \overline{E} = \sigma_s/\varepsilon_o$, $\hat{n}$ is the surface normal, and σ_s is the surface charge density. The surface of each conductor is an equipotential surface, as are all non-physical surfaces that are locally perpendicular to $\overline{E}$. Figure R11-2 illustrates a flux tube between two charged surfaces at electric potentials that differ by V volts. Only some of the field lines are illustrated.

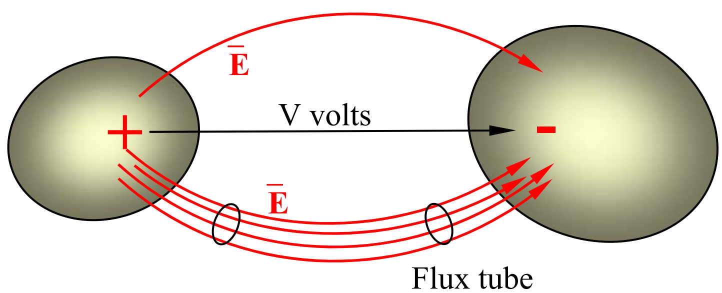

Figure R11-2. Electric flux tube between charged surfaces

Flux tubes are bundles of electric (or magnetic) field lines in charge-free regions; they are divergence-free by Gauss's laws. The integral form of Gauss's law when $\rho = 0$ is:

$$\int_V \rho\, dv = \int_S \hat{n} \bullet \varepsilon \overline{E}\, da = 0 \quad (12)$$

[5] See (4.1.19-20) on page 123 in the text.

and it requires that the total flux entering a flux tube at one end must equal that exiting at the other end because no flux exits the sidewalls; the sidewalls are parallel to the field lines by definition. Electric flux is defined by (12) for each end of the flux tube.

This visual metaphor is best seen in a two-dimensional context, such as illustrated in Figure R11-3, where the given cross-section of the two conductors is constant and infinitely extended in the z direction. This example has the form of a parallel-plate capacitor near the center, and then opens up into more arbitrary geometries on the ends. The integral of $\overline{E}$ from the top surface to the bottom surface along any arbitrary path is always V volts, because the top and bottom conductors are each equipotential objects. The equipotential surfaces are parallel to the conductors and perpendicular to the field lines.

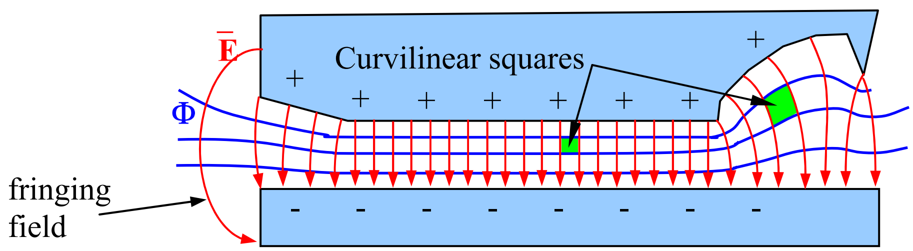

Figure R11-3. Graphical field mapping of $\overline{E}$ and F between charged conductors

One approach to *graphical field mapping* begins by guessing and sketching the equipotential surfaces, typically starting with that potential midway between those of the two conductors. Then this divide-by-two strategy can be repeated, yielding the three equipotential surfaces illustrated in Figure R11-3. The next step is to add the electric field lines $\overline{E}$, beginning where they can be guessed more accurately. They should be everywhere perpendicular to the equipotential surfaces and form curvilinear squares with them, as illustrated. Each curvilinear square can then be divided into quarters by adding mid-square electric field lines and equipotentials. The uniquely correct field solution is that for which these squares approach perfection as this divide-by-two algorithm is continued indefinitely. Pencil sketches using this algorithm typically converge quickly, with only a few erasures before a useful approximate solution is achieved. In the illustrated case, it is clear that the field lines $\overline{E}$ become more intense (closer together) near sharp points, and less intense where the gap is greater. Thus strong electric fields are often found near sharp points.

Graphical field mapping works for this two-dimensional case because the resulting field solution satisfies all of Maxwell's equations: 1) the resulting field $\overline{E}$ is divergence and curl free and perpendicular to the equipotentials and bounding conductors, and 2) the local distances between adjacent equipotentials and field lines are both inversely proportional to field strength, thus enhancing the squareness of the curvilinear squares as they are increasingly subdivided. A similar mapping technique, not discussed here, can be used to determine magnetic field lines and magnetic equipotentials in magnetic structures.

A more general and exact way to determine electric fields near conducting structures is to solve for the scalar potential Φ using *Laplace's equation*:

$$\nabla^2\Phi = 0 \tag{13}$$

which follows from inserting (1) into Gauss's law for a charge-free region, $\nabla \bullet \varepsilon \overline{E} = 0$, and recalling the vector identity $\nabla \bullet \nabla\Phi = 0$. Standard simple solutions to Laplace's equation have been developed for rectangular, cylinderical, and spherical coordinate systems, and numerical methods have been developed for arbitrary configurations. The nature of these solutions is suggested by the following example for two-dimensional rectangular coordinates. In these coordinates Laplace's equation (13) becomes:

$$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = 0 \tag{14}$$

We now use a technique called *separation of variables* and postulate solutions of the form:

$$\Phi(x,y) = X(x)Y(y) \tag{15}$$

Substituting (15) into (14) and then dividing by X(x)Y(y) yields:

$$[d^2X(x)/dx^2]/X(x) = -[d^2Y(y)/dy^2]/Y(y) \tag{16}$$

We now note that the only way (16) can be satisfied everywhere is for each side of the equation to equal a constant, defined as k^2. This constant is called the *separation constant* because (16) can be separated into two independent differential equations that can be solved separately:

$$d^2X(x)/dx^2 = -k^2X \qquad d^2Y(y)/dy^2 = k^2Y \tag{17}$$

When k = 0 the generic solutions for these two equations are:

$$X(x) = Ax + B \qquad Y(y) = Cy + D \tag{18}$$

and when $k \neq 0$:

$$X(x) = A\cos kx + B\sin kx \qquad Y(y) = C\cosh ky + D\sinh ky \tag{19}$$

as can be easily seen by substituting (18) or (19) into (17). Note that by letting $k \rightarrow jk$, the sinusoidal x-dependence may be transformed into a hyperbolic x-dependence, and vice-versa for the y-dependence; one axis must always vary sinusoidally when the other has a hyperbolic form. The total solution for Φ when k = 0 might then be $\Phi(x,y) = (A + Bx)(C + Dy)$. The constants in these generic solutions (k, A, B, C, and D) are chosen so that Φ matches the boundary conditions. That is, the unknown constants are chosen so that (15) yields a solution Φ that equals the potentials on the boundaries as specified in the problem. Obviously this strongly limits the number of problems that can be solved this way. Similarly simple solutions exist for other coordinate systems, but this approach will not be pursued further here.

Now we are in a position to analyze the fields in resistors, capacitors, and inductors and similar devices that are quasistatic, i.e. very small compared to a wavelength. Consider first the parallel-plate structure illustrated in Figure R11-4, where the conducting plates ($\sigma_{plates} = \infty$) have area A and separation d, and the space between is filled by a medium characterized by ε, μ, and conductivity σ. First let the medium be insulating ($\sigma = 0$) and $d << A^{0.5}$ so that the fringing fields at the edges can be neglected. Then $\overline{E}$ must be perpendicular to the plates and both divergence and curl free, which means that it must be uniform and of value E_o [vm^{-1}]. If the potential difference between the plates is V volts, then the integral form of (1) becomes:

$$V = \Phi_1 - \Phi_2 = \int_1^2 \overline{E} \bullet d\overline{s} = E_o d \qquad (20)$$

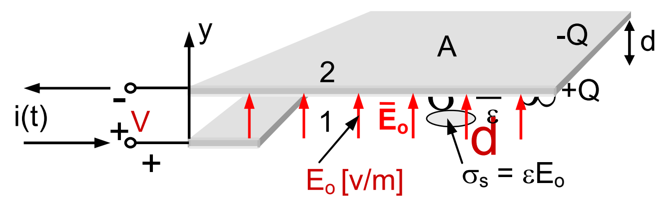

Figure R11-4. Capacitor

The total charge Q on each plate can be found by integrating the surface charge density σ_s over the plate area A; $Q = A\sigma_s$ where σ_s follows from Gauss's law and its associated boundary condition at a perfect conductor, $\hat{n} \bullet \overline{E} = \sigma_s/\varepsilon$. Thus $\sigma_s = \varepsilon E_o$ and:

$$Q = A\varepsilon E_o = A\varepsilon V/d = CV \text{ [Coulombs]} \qquad (21)$$

We define the *capacitance* C [Farads] of a device as $C = Q/V$, or $Q = CV$. Solving (21) for C yields the approximate capacitance for a *parallel-plate capacitor*:

$$C = A\varepsilon/d \text{ [F]} \qquad (22)$$

Its I-V behavior can be found by noting $i(t) = dQ(t)/dt$, so that (21) yields the current:

$$i(t) = C\, dV(t)/dt \qquad (23)$$

The same Figure R11-4 applies to resistors, where the medium now has conductivity[6] $\sigma > 0$. In this case we note that the uniform current density is $\overline{J} = \sigma \overline{E}_o$ and $E_o = V/d$, so the total current I is:

$$I = \overline{J}A = A\sigma V/d = V/R \text{ [Amperes]} \qquad (24)$$

[6] The symbols σ for conductivity [siemens m^{-1}] and surface charge σ_s [coulombs m^{-2}] should not be confused.

where the *resistance* R is:

$$R = d/\sigma A \quad [\text{Ohms}] \tag{25}$$

Finally, it is useful to note that the electric energy w_e [J] stored in a capacitor is simply related to C and its voltage V. We can find w_e by integrating the electrical power into the capacitor as it transitions from its rest state (zero energy and voltage) to some final state at time t:

$$w_e = \int_o^t v(t)\, i(t)\, dt = C\int_o^V v(t)\, dv = CV^2/2 \quad [\text{J}] \tag{26}$$

where we used (23) to substitute for i(t), and converted the integral to one over voltage.