6.013, Electromagnetic Fields, Forces, and Motion
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Lecture 4: The Scalar Electric Potential and the Coulomb Superposition Integral

I. Quasistatics

Electroquasistatics (EQS)

$$\nabla \times \overline{E} = -\cancelto{0}{\frac{\partial}{\partial t}\left(\mu_0 \overline{H}\right)} \approx 0$$

$$\nabla \bullet \left(\varepsilon_0 \overline{E}\right) = \rho$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial}{\partial t}\left(\varepsilon_0 \overline{E}\right)$$

$$\nabla \bullet \overline{J} + \frac{\partial \rho}{\partial t} = 0$$

Magnetquasistatics (MQS)

$$\nabla \times \overline{E} = -\frac{\partial}{\partial t}\left(\mu_0 \overline{H}\right)$$

$$\nabla \times \overline{H} = \overline{J} + \cancelto{0}{\frac{\partial}{\partial t}\left(\varepsilon_0 \overline{E}\right)}$$

$$\nabla \bullet \left(\mu_0 \overline{H}\right) = 0$$

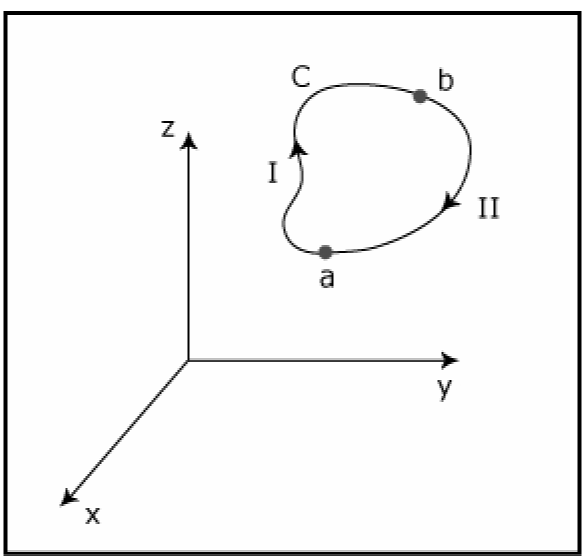
$$\nabla \bullet \overline{J} = 0$$

$$\nabla \bullet \left(\varepsilon_0 \overline{E}\right) = \rho$$

II. Irrotational EQS Electric Field

1. Conservative Electric Field

$$\oint_C \overline{E} \bullet \overline{ds} = -\cancelto{0}{\frac{d}{dt}\int_S \mu_0 \overline{H} \bullet \overline{da}} \approx 0$$



$$\oint_C \overline{E} \bullet \overline{ds} = \underset{\mathrm{I}}{\int_a^b} \overline{E} \bullet \overline{ds} + \underset{\mathrm{II}}{\int_b^a} \overline{E} \bullet \overline{ds} = 0 \Rightarrow \underbrace{\underset{\mathrm{I}}{\int_a^b} \overline{E} \bullet \overline{ds}}_{\substack{\text{Electromotive Force} \\ \text{(EMF)}}} = \underset{\mathrm{II}}{\int_a^b} \overline{E} \bullet \overline{ds}$$

EMF between 2 points (a, b) independent of path
$\overline{E}$ field is <u>conservative</u>

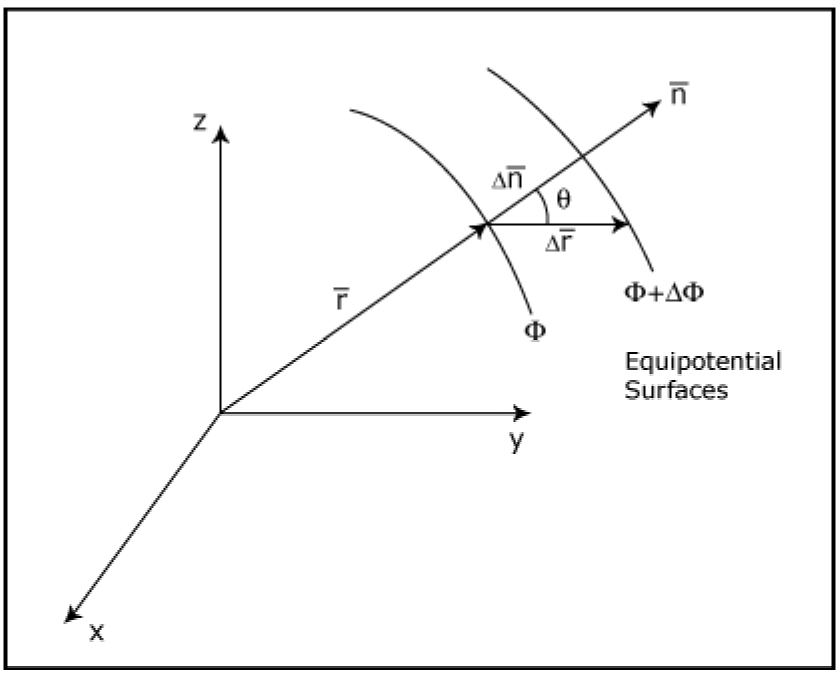
$$\Phi(\overline{r}) - \Phi(\overline{r}_{ref}) = \int_{\overline{r}}^{\overline{r}_{ref}} \overline{E} \bullet \overline{ds}$$

↗ Scalar

electric potential

$$\int_a^b \overline{E} \bullet \overline{ds} = \int_a^{\overline{r}_{ref}} \overline{E} \bullet \overline{ds} + \int_{\overline{r}_{ref}}^b \overline{E} \bullet \overline{ds} = \Phi(a) - \Phi(\overline{r}_{ref}) + \Phi(\overline{r}_{ref}) - \Phi(b) = \Phi(a) - \Phi(b)$$

2. The Electric Scalar Potential



$$\overline{r} = x\, \overline{i}_x + y\, \overline{i}_y + z\, \overline{i}_z$$

$$\Delta\overline{r} = \Delta x\, \overline{i}_x + \Delta y\, \overline{i}_y + \Delta z\, \overline{i}_z$$

$$\Delta n = \Delta r \cos\theta$$

$$\Delta\Phi = \Phi\left(x+\Delta x, y+\Delta y, z+\Delta z\right) - \Phi\left(x,y,z\right)$$

$$= \Phi\left(x,y,z\right) + \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z - \Phi\left(x,y,z\right)$$

$$= \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z$$

$$= \underbrace{\left[\frac{\partial\Phi}{\partial x}\bar{i}_x + \frac{\partial\Phi}{\partial y}\bar{i}_y + \frac{\partial\Phi}{\partial z}\bar{i}_z\right]}_{\text{grad } \Phi = \nabla\Phi} \bullet \Delta\bar{r}$$

$$\nabla = \bar{i}_x\frac{\partial}{\partial x} + \bar{i}_y\frac{\partial}{\partial y} + \bar{i}_z\frac{\partial}{\partial z}$$

$$\text{grad } \Phi = \nabla\Phi = \bar{i}_x\frac{\partial\Phi}{\partial x} + \bar{i}_y\frac{\partial\Phi}{\partial y} + \bar{i}_z\frac{\partial\Phi}{\partial z}$$

$$\int_{\bar{r}}^{\bar{r}+\Delta\bar{r}} \bar{E}\bullet\overline{ds} = \Phi\left(\bar{r}\right) - \Phi\left(\bar{r}+\Delta\bar{r}\right) = -\Delta\Phi = -\nabla\Phi\bullet\Delta\bar{r} = \bar{E}\bullet\Delta\bar{r}$$

$$\boxed{\bar{E} = -\nabla\Phi}$$

$$\Delta\Phi = \frac{\Delta\Phi}{\Delta n}\Delta r\cos\theta = \frac{\Delta\Phi}{\Delta n}\bar{n}\bullet\Delta\bar{r} = \nabla\Phi\bullet\Delta\bar{r}$$

$$\nabla\Phi = \frac{\Delta\Phi}{\Delta n}\bar{n} = \frac{\partial\Phi}{\partial n}\bar{n}$$

The gradient is in the direction perpendicular to the equipotential surfaces.

III. Vector Identity

$$\nabla\times\bar{E} = 0$$

$$\bar{E} = -\nabla\Phi$$

$$\nabla\times\left(\nabla\Phi\right) = 0$$

IV. Sample Problem

$$\Phi(x,y) = \frac{V_0 xy}{a^2} \qquad \text{(Equipotential lines hyperbolas: xy=constant)}$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial x}\bar{i}_x + \frac{\partial\Phi}{\partial y}\bar{i}_y\right]$$

$$= \frac{-V_0}{a^2}\left(y\,\bar{i}_x + x\,\bar{i}_y\right)$$

Electric Field Lines [lines tangent to electric field]

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{y} \Rightarrow ydy = xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$y^2 - x^2 = y_0^2 - x_0^2$ [lines pass through point (x_0, y_0)]
(hyperbolas orthogonal to xy)

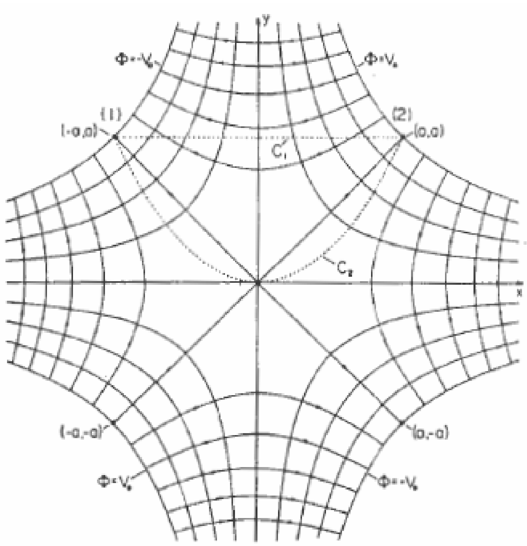


Figure 4.1.3 Cross-sectional view of surfaces of constant potential for two-dimensional potential given by (18).

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

V. Poisson's Equation

$$\nabla \cdot \overline{E} = \nabla \cdot (-\nabla\Phi) = \rho/\varepsilon_0 \Rightarrow \nabla^2\Phi = -\rho/\varepsilon_0$$

$$\nabla^2\Phi = \nabla \cdot (\nabla\Phi) = \left[\overline{i}_x \frac{\partial}{\partial x} + \overline{i}_y \frac{\partial}{\partial y} + \overline{i}_z \frac{\partial}{\partial z}\right] \cdot \left[\frac{\partial\Phi}{\partial x}\overline{i}_x + \frac{\partial\Phi}{\partial y}\overline{i}_y + \frac{\partial\Phi}{\partial z}\overline{i}_z\right]$$

$$= \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

VI. Coulomb Superposition Integral

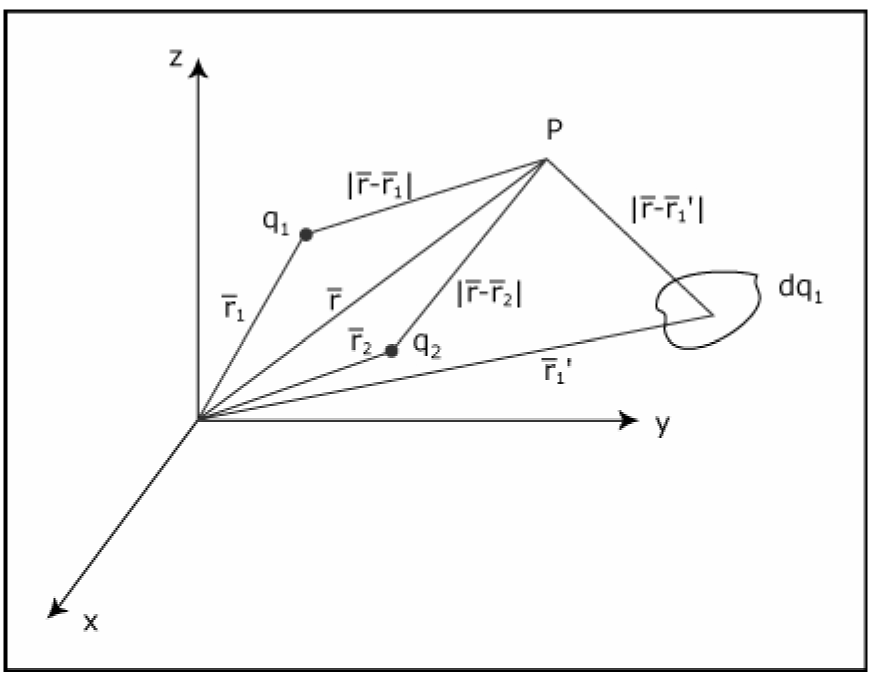
1. Point Charge

$$E_r = -\frac{\partial\Phi}{\partial r} = \frac{q}{4\pi\varepsilon_0 r^2} \Rightarrow \Phi = \frac{q}{4\pi\varepsilon_0 r} + C$$

Take reference $\Phi(r \to \infty) = 0 \Rightarrow C = 0$

$$\Phi = \frac{q}{4\pi\varepsilon_0 r}$$

2. Superposition of Charges



$$d\Phi_T(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{\left|\bar{r}-\bar{r}_1\right|} + \frac{q_2}{\left|\bar{r}-\bar{r}_2\right|} + \ldots \quad \frac{dq_1}{\left|\bar{r}-\bar{r}'_1\right|} + \frac{dq_2}{\left|\bar{r}-\bar{r}'_2\right|} + \ldots\right]$$

$$\Phi_T(P) = \frac{1}{4\pi\varepsilon_0}\left[\sum_{n=1}^{N}\frac{q_n}{\left|\bar{r}-\bar{r}_n\right|} + \int\limits_{\substack{\text{all line,}\\ \text{surface, and}\\ \text{volume charges}}} \frac{dq}{\left|\bar{r}-\bar{r}'\right|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\sum_{n=1}^{N}\frac{q_n}{\left|\bar{r}-\bar{r}_n\right|} + \int_L \frac{\lambda\left(\bar{r}'\right)dl'}{\left|\bar{r}-\bar{r}'\right|} + \int_S \frac{\sigma_s\left(\bar{r}'\right)da'}{\left|\bar{r}-\bar{r}'\right|} + \int_V \frac{\rho\left(\bar{r}'\right)dV'}{\left|\bar{r}-\bar{r}'\right|}\right]$$

Short-hand notation

$$\Phi\left(\bar{r}\right) = \int_V \frac{\rho\left(\bar{r}'\right)dV'}{4\pi\varepsilon_0\left|\bar{r}-\bar{r}'\right|}$$