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6.013 Electromagnetics and Applications, Fall 2005

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which is plotted versus  $kL$  in Fig. 9-14. This result can be checked in the limit as  $L$  becomes very small ( $kL \ll 1$ ) since the radiation resistance should approach that of a point dipole given in Section 9-2-5. In this short dipole limit the bracketed terms in (14) are

$$\lim_{kL \ll 1} \left\{ \begin{array}{l} \frac{\sin kL}{kL} \approx 1 - \frac{(kL)^2}{6} \\ \cos kL \approx 1 - \frac{(kL)^2}{2} \\ kL \text{Si}(kL) \approx (kL)^2 \end{array} \right. \quad (15)$$

so that (14) reduces to

$$\lim_{kL \ll 1} R \approx \frac{\eta}{2\pi} \frac{(kL)^2}{3} = \frac{2\pi\eta}{3} \left(\frac{L}{\lambda}\right)^2 = 80\pi^2 \left(\frac{L}{\lambda}\right)^2 \sqrt{\frac{\mu_r}{\epsilon_r}} \quad (16)$$

which agrees with the results in Section 9-2-5. Note that for large dipoles ( $kL \gg 1$ ), the sine integral term dominates with  $\text{Si}(kL)$  approaching a constant value of  $\pi/2$  so that

$$\lim_{kL \gg 1} R \approx \frac{\eta kL}{4} = 60 \sqrt{\frac{\mu_r}{\epsilon_r}} \pi^2 \frac{L}{\lambda} \quad (17)$$

## PROBLEMS

### Section 9-1

1. We wish to find the properties of waves propagating within a linear dielectric medium that also has an Ohmic conductivity  $\sigma$ .

- (a) What are Maxwell's equations in this medium?
- (b) Defining vector and scalar potentials, what gauge condition decouples these potentials?
- (c) A point charge at  $r = 0$  varies sinusoidally with time as  $Q(t) = \text{Re}(\dot{Q} e^{j\omega t})$ . What is the scalar potential?
- (d) Repeat (a)–(c) for waves in a plasma medium with constitutive law

$$\frac{\partial \mathbf{J}_f}{\partial t} = \omega_p^2 \epsilon \mathbf{E}$$

2. An infinite current sheet at  $z = 0$  varies as  $\text{Re}[K_0 e^{j(\omega t - k_x x)} \mathbf{i}_x]$ .

- (a) Find the vector and scalar potentials.
- (b) What are the electric and magnetic fields?

(c) Repeat (a) and (b) if the current is uniformly distributed over a planar slab of thickness  $2a$ :

$$\mathbf{J}_f = \begin{cases} J_0 e^{j(\omega t - k_z z)} \mathbf{i}_x, & -a < z < a \\ 0, & |z| > a \end{cases}$$

3. A sphere of radius  $R$  has a uniform surface charge distribution  $\sigma_f = \text{Re}(\hat{\sigma}_0 e^{j\omega t})$  where the time varying surface charge is due to a purely radial conduction current.

(a) Find the scalar and vector potentials, inside and outside the sphere. (**Hint:**  $r_{QP}^2 = r^2 + R^2 - 2rR \cos \theta$ ;  $r_{QP} dr_{QP} = rR \sin \theta d\theta$ .)

(b) What are the electric and magnetic fields everywhere?

### Section 9.2

4. Find the effective lengths, radiation resistances and line charge distributions for each of the following current distributions valid for  $|z| < dl/2$  on a point electric dipole with short length  $dl$ :

(a)  $\hat{I}(z) = I_0 \cos \alpha z$

(b)  $\hat{I}(z) = I_0 e^{-\alpha|z|}$

(c)  $\hat{I}(z) = I_0 \cosh \alpha z$

5. What is the time-average power density, total time-average power, and radiation resistance of a point magnetic dipole?

6. A plane wave electric field  $\text{Re}(\mathbf{E}_0 e^{j\omega t})$  is incident upon a perfectly conducting spherical particle of radius  $R$  that is much smaller than the wavelength.

(a) What is the induced dipole moment? (**Hint:** See Section 4-4-3.)

(b) If the small particle is, instead, a pure lossless dielectric with permittivity  $\epsilon$ , what is the induced dipole moment?

(c) For both of these cases, what is the time-average scattered power?

7. A plane wave magnetic field  $\text{Re}(\mathbf{H}_0 e^{j\omega t})$  is incident upon a perfectly conducting particle that is much smaller than the wavelength.

(a) What is the induced magnetic dipole moment? (**Hint:** See Section 5-7-2ii and 5-5-1.)

(b) What are the re-radiated electric and magnetic fields?

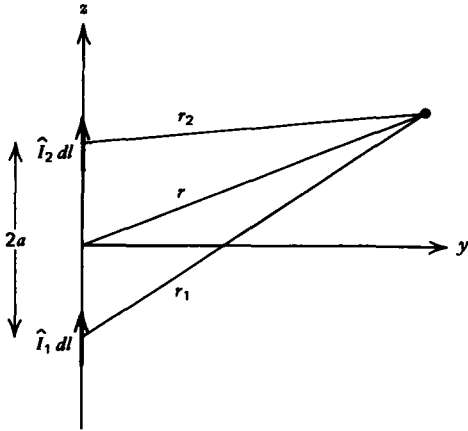
(c) What is the time-average scattered power? How does it vary with frequency?

8. (a) For the magnetic dipole, how are the magnetic field lines related to the vector potential  $\mathbf{A}$ ?

(b) What is the equation of these field lines?

### Section 9.3

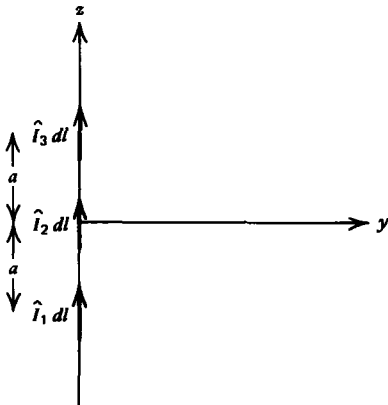
9. Two aligned dipoles  $\hat{I}_1 dl$  and  $\hat{I}_2 dl$  are placed along the  $z$  axis a distance  $2a$  apart. The dipoles have the same length



while the currents have equal magnitudes but phase difference  $\chi$ .

- (a) What are the far electric and magnetic fields?
- (b) What is the time-average power density?
- (c) At what angles is the power density zero or maximum?
- (d) For  $2a = \lambda/2$ , what values of  $\chi$  give a broadside or end-fire array?
- (e) Repeat (a)–(c) for  $2N + 1$  equally spaced aligned dipoles along the  $z$  axis with incremental phase difference  $\chi_0$ .

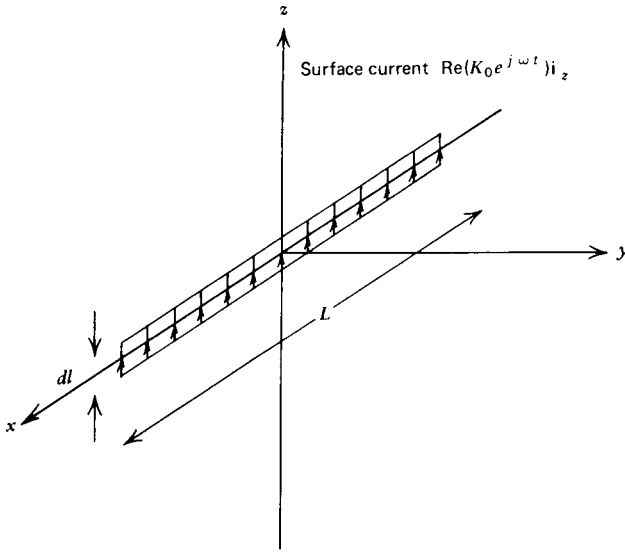
10. Three dipoles of equal length  $dl$  are placed along the  $z$  axis.



- (a) Find the far electric and magnetic fields.
- (b) What is the time average power density?
- (c) For each of the following cases find the angles where the power density is zero or maximum.

- (i)  $\hat{I}_1 = \hat{I}_3 = I_0, \hat{I}_2 = 2I_0$
- (ii)  $\hat{I}_1 = \hat{I}_3 = I_0, \hat{I}_2 = -2I_0$
- (iii)  $\hat{I}_1 = -\hat{I}_3 = I_0, \hat{I}_2 = 2jI_0$

11. Many closely spaced point dipoles of length  $dl$  placed along the  $x$  axis driven in phase approximate a  $z$ -directed current sheet  $\text{Re}(K_0 e^{j\omega t} \mathbf{i}_z)$  of length  $L$ .



- (a) Find the far fields from this current sheet.  
 (b) At what angles is the power density minimum or maximum?

#### Section 9.4

12. Find the far fields and time-average power density for each of the following current distributions on a long dipole:

$$(a) \hat{I}(z) = \begin{cases} I_0(1 - 2z/L), & 0 < z < L/2 \\ I_0(1 + 2z/L), & -L/2 < z < 0 \end{cases}$$

**Hint:**

$$\int z e^{az} dz = \frac{e^{az}}{a^2} (az - 1)$$

$$(b) \hat{I}(z) = I_0 \cos \pi z/L, \quad -L/2 < z < L/2$$

**Hint:**

$$\int e^{az} \cos pz dz = e^{az} \frac{(a \cos pz + p \sin pz)}{(a^2 + p^2)}$$

- (c) For these cases find the radiation resistance when  $kL \ll 1$ .