Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Recitation 23 - Solutions
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1. (a) $E(T_n) = E(A_n) = 0$, $\text{var}(A_n) = \frac{16}{n}$, $\text{var}(T_n) = 16n$

   (b) By the WLLN, A_n is stochastically convergent to 0.

   $$\lim_{n\to\infty} \mathbf{P}(|A_n - 0| \geq \epsilon) = 0.$$

   T_n is not stochastically convergent. Since the pmf of T_n is symmetric around 0, if T_n is convergent in probability, it should converge to 0. However,

   $$\begin{aligned} & \lim_{n\to\infty} \mathbf{P}(|T_n - 0| \geq \epsilon) \\ \geq\ & \lim_{n\to\infty} \mathbf{P}(T_n \geq \epsilon) \\ =\ & \lim_{n\to\infty} \mathbf{P}(\frac{T_n}{4\sqrt{n}} \geq \frac{\epsilon}{4\sqrt{n}}) \\ \geq\ & \lim_{n\to\infty} \mathbf{P}(\frac{T_n}{4\sqrt{n}} \geq \frac{\epsilon}{4}) \\ =\ & \Phi(-\frac{\epsilon}{4}) > 0 \quad \text{(by the CLT)}. \end{aligned}$$

   Hence, T_n is not convergent in probability.

   (c) Since n is quite large, let us apply the CLT, which gives $Q = \mathbf{P}(\frac{|T_{100}-\mathbf{E}[T_{100}]|}{\sigma} \geq \frac{32}{\sigma}) = 2 * \Phi(-\frac{32}{\sigma})$ (σ is 40). This is equal to $2 * \Phi(-0.8)$, or 0.4237.

   We may improve the approximation by using $Q = \mathbf{P}(\frac{|T_{100}-\mathbf{E}[T_{100}]|}{\sigma} \geq \frac{32-8\times\frac{1}{2}}{40}) = 2 * \Phi(-0.7)$, which will give $2 * \Phi(-0.7)$, or 0.4839. However, the second approximation is not necessary to be better.

2. The 64th-order erlang random variable is the sum of 64 i.i.d. exponential random variables ($\lambda = 3$), against which a CLT approximation is good around the mean value, which is 64/3. The approximating Gaussian PDF at the same mean (64/3) and variance (64/9) is

   $$p_Y(y) = \frac{1}{\sqrt{2\pi}\frac{8}{3}} e^{-\frac{(y-\frac{64}{3})^2}{\frac{128}{9}}},$$

   and when we substitute 68/3 for y, we conclude $a = \frac{8}{3}$ and $b = \pm\frac{4}{3}$.

3. Please see online solution for Problem 7.7.