Problem Set 5G: Solutions
Due: March 16, 2005

G1. When finding the probability density function of a function of random variables, we first need to find the cumulative distribution function, and then take the necessary partial derivatives to find the density function.

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(\sqrt{2B}\cos A \leq x) \\ &= \iint_{\sqrt{2B}\cos A \leq x} \frac{1}{2\pi} e^{-B} dB dA. \end{aligned}$$

This integral is most easily accomplished by the change of variables formula. The change of variables formula is the higher dimensional equivalent to substitution in single variable calculus. The result is available in any advanced calculus textbook, and the proof is available in most of the rigorous advanced calculus text books (e.g. see Rudin). It will be easiest to compute the joint distribution of X, Y using the change of variables formula.

$$\begin{aligned} F_{X,Y}(x,y) &= P(X \leq x, Y \leq y) \\ &= P(\sqrt{2B}\cos A \leq x, \sqrt{2B}\sin A \leq y) \\ &= \iint_{\sqrt{2B}\cos A \leq x, \sqrt{2B}\sin A \leq y} \frac{1}{2\pi} e^{-B} dB dA \\ &= \iint_{X \leq x, Y \leq y} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dx dy \end{aligned}$$

Taking the partial derivatives with respect to X and Y we immediately recognize this as the bivariate distribution of two independent, standard normal random variables. To convince ourselves of this, we can easily find the marginal densities for X and Y, to see that indeed,

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y).$$