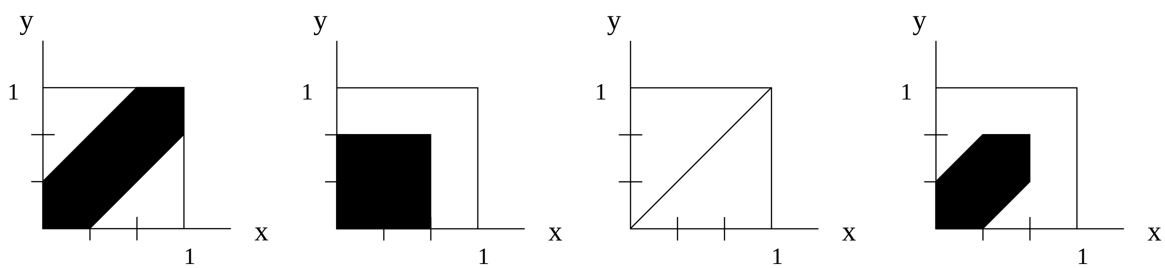
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Problem Set 1
Due: February 9, 2005

1. The events A, B, $A \cap B$, and C are shown in the shaded areas of the sample spaces below. The physical description of the events given in the problem can be related to mathematical inequalities to define certain regions of the sample space.. Be sure that you are able to match regions of the sample space with the descriptions of the events. Note that the event AB is an intersection of the region for A and B.



A: {| x - y | <= 1/3}　**B : {x <= 2/3, y <= 2/3}**　**C : {x = y}**　**A ∩ B**

Since we are dealing with a "uniform probability law", we know that probability is proportional to the area of the shaded regions. Since the whole space is the unit square, and the unit square has area equal to one, the probability of any event is exactly the area of its respective region. (If the whole space were a square of different size, we would need a normalizing factor.)

Thus, finding $P(A)$, $P(B)$, $P(A \cap B)$, and $P(C)$ amounts to finding the area of the regions. These turn out to be

$$P(A) = \frac{5}{9}$$
$$P(B) = \frac{4}{9}$$
$$P(A \cap B) = \frac{1}{3}$$
$$P(C) = 0$$

Note that $P(C)$ is zero. Continuous random variables tend to have this peculiar problem when dealing with exact values. This will be explored later.

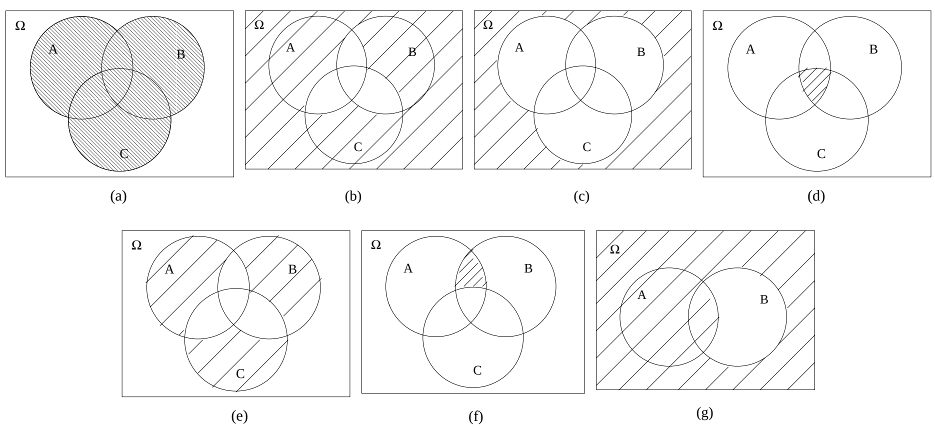
2. (a) Since the cars are all distinct, there are 20! ways to line them all up.

   (b) To find the probability that the cars will be parked in such a way that they will be alternating: US made, foreign made, etc... we will count the number of "favorable" outcomes, and divide by the total number of outcomes which we found in part (a) above. We count in the following manner: first lay the US cars down. We can do this in 10! ways, since the cars are distinct. Now lay the foreign cars in-between the US cars. Again we can do this in 10! ways. Finally, we need to multiply by 2, since the sequence could begin either with a US car or with a foreign car. Thus we have a total of $2 \cdot 10! \cdot 10!$, and the final answer is

$$\frac{2 \cdot 10! \cdot 10!}{20!}.$$

Note that we could have solved the second part of the problem by neglecting the fact that the cars are distinct. Suppose that the foreign cars are indistinguishable, and also suppose that the US cars are indistinguishable. Again we count the number of "favorable" outcomes in the same way: lay the US cars down in one way. Then there are two ways to lay the foreign cars down since the sequence can begin with either a US or a foreign car. Thus there are two favorable outcomes, out of a possible $\frac{20!}{10! \cdot 10!}$, and the two methods yield the same answer.

3. (a) $A \cup B \cup C$
   (b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$
   (c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$
   (d) $A \cap B \cap C$
   (e) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
   (f) $A \cap B \cap C^c$
   (g) $A \cup (A^c \cap B^c)$



4. (a) From $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and the nonnegativity axiom applied to $P(A \cap B)$, we obtain $P(A \cup B) \leq P(A) + P(B)$, which proves the first part of the inequality. To prove the second part, note that $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$. Therefore, $P(A) \leq P(A \cup B)$ and $P(B) \leq P(A \cup B)$ i.e., $P(A \cup B) \geq \max\{P(A), P(B)\}$.

   The first inequality is satisfied with equality when the events A and B are disjoint. The second inequality is satisfied with equality when one of the sets A or B is completely included in the other i.e., $A \subseteq B$ or $B \subseteq A$.

   (b) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $P(A) \geq P(A \cap B)$ and $P(B) \geq P(A \cap B)$. Therefore, $\min\{P(A), P(B)\} \geq P(A \cap B)$, which proves the first part of the inequality. The second part can be derived in the following way:

   (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (probability law property)
   (ii) $P(A \cup B) + P((A \cup B)^c) = 1$ (normalization axiom)
   (iii) $P((A \cup B)^c) \geq 0$ (nonnegativity axiom)

Expressions (ii) and (iii) together imply $P(A \cup B) \leq 1$. Combining with expression (i), we have $P(A)+P(B)-P(A\cap B) \leq 1$ that can be arranged to conclude the desired result, $P(A) + P(B) - 1 \leq P(A \cap B)$. Note that this is Bonferroni's inequality, as discussed in recitation and problem 1.8 of the text.

The first inequality is satisfied with equality when $A \subseteq B$ or $B \subseteq A$, and the second inequality is satisfied with equality when $P(A \cup B) = 1$, or the union $A \cup B$ represents the whole sample space.

5. We begin by enumerating the sample space Ω and identifying the *relative* probabilities of all outcomes, as shown in the table below, where $p \in [0, 1]$ will need to be determined.

Die 1	Die 2	Product	P(Product)
1	1	1	p
1	2	2	2p
1	3	3	3p
1	4	4	4p
2	1	2	2p
2	2	4	4p
2	3	6	6p
2	4	8	8p
3	1	3	3p
3	2	6	6p
3	3	9	9p
3	4	12	12p
4	1	4	4p
4	2	8	8p
4	3	12	12p
4	4	16	16p
		Total	100p

$$P(\Omega) = 1 = 100p \quad \Rightarrow \quad p = \frac{1}{100} = 0.01$$

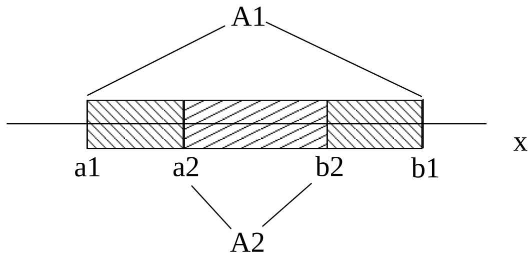
(a) Let set A indicate the event that the product is even. Then,

$$P(A) = 2p + 4p + 2p + 4p + 6p + 8p + 6p + 12p + 4p + 8p + 12p + 16p = 84p = \boxed{0.84}$$

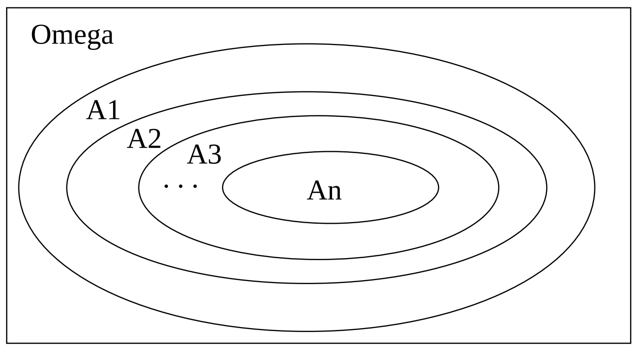
(b) Let set B indicate the event of rolling a 2 and 3. Then,

$$P(B) = P(2, 3) + P(3, 2) = 6p + 6p = 12p = \boxed{0.12}$$

G1†. First, consider the set $A_1 = \{x \mid a_1 \leq x \leq b_1\}$ and the set $A_2 = \{x \mid a_2 \leq x \leq b_2\}$. Since a_n is an increasing sequence, $a_1 \leq a_2$ and since b_n is a decreasing sequence, $b_1 \geq b_2$. As we see in the following diagram, we have $A_2 \subset A_1$.



Continuing this argument up to A_n, we get the following picture:



Finally, $A_\infty = \{x \mid a \leq x \leq b\}$. So, by the above picture, $A_\infty = \lim_{n\to\infty} A_n = (\bigcup_{i=1}^{\infty} A_i^c)^c$. Observe that $A_n, n \geq 1$ is a decreasing sequence and that $A_n^c, n \geq 1$ is an increasing sequence. Now, we define the events $B_n, C_n, n \geq 1$ as follows:

$$\begin{aligned} C_n &= A_n^c \\ \\ B_1 &= C_1 \\ B_2 &= C_2 \cap C_1^c \\ B_n &= C_n \cap C_{n-1}^c \end{aligned}$$

Thus, each B_n consists of elements that are not in the previous events and are consequently mutually exclusive. Furthermore:

$$\begin{aligned} \bigcup_{i=1}^{n} B_i &= C_1 \bigcup (C_2 \cap C_1^c) \bigcup ... \bigcup (C_n \cap C_{n-1}^c) \\ \bigcup_{i=1}^{n} B_i &= \bigcup_{i=1}^{n} C_i \\ \bigcup_{i=1}^{\infty} B_i &= \bigcup_{i=1}^{\infty} C_i \end{aligned}$$

So, by additivity,

$$\begin{aligned}
P(\bigcup_{i=1}^{\infty} C_i) &= P(\bigcup_{i=1}^{\infty} B_i) = \Sigma_{i=1}^{\infty} P(B_i) \\
&= \lim_{n\to\infty} \Sigma_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} B_i) \\
&= \lim_{n\to\infty} P(\bigcup_{i=1}^{n} C_i) = \lim_{n\to\infty} P(C_n) \\
&= \lim_{n\to\infty} P(A_n^c)
\end{aligned}$$

Also:

$$P(\bigcup_{i=1}^{\infty} C_i) = P(\bigcup_{i=1}^{\infty} A_i^c) = P[((\bigcup_{i=1}^{\infty} A_i^c)^c)^c]$$

$$\begin{aligned}
P[((\bigcup_{i=1}^{\infty} A_i^c)^c)^c] &= 1 - P[(\bigcup_{i=1}^{\infty} A_i^c)^c] = 1 - P(A_\infty) \\
\lim_{n\to\infty} P(A_n^c) &= \lim_{n\to\infty} (1 - P(A_n)) = 1 - \lim_{n\to\infty} P(A_n)
\end{aligned}$$

Therefore:

$$\begin{aligned}
1 - P(A_\infty) &= 1 - \lim_{n\to\infty} P(A_n) \\
P(A_\infty) &= \lim_{n\to\infty} P(A_n) \\
P(\{x \mid a \le x \le b\}) &= \lim_{n\to\infty} P(\{x \mid a_n \le x \le b_n\}) \\
P([a,b]) &= \lim_{n\to\infty} P([a_n, b_n])
\end{aligned}$$