MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Tutorial 1 Answers
Week of February 7, 2005

1. Yes. Since $A \subset B$, $\mathbf{P}(A \cap B) = \mathbf{P}(A)$. This is equal to $\mathbf{P}(A) \cdot \mathbf{P}(B)$ when $\mathbf{P}(B) = 1$ or $\mathbf{P}(A) = 0$.

2. **(THE CHESS PROBLEM)**

   (a) i. $\mathbf{P}(\text{2nd Rnd Req}) = (0.6)^2 + (0.4)^2 = 0.52$
   ii. $\mathbf{P}(\text{Bo Wins 1st Rnd}) = (0.6)^2 = 0.36$
   iii. $\mathbf{P}(\text{Al Champ}) = 1 - \mathbf{P}(\text{Bo Champ}) - \mathbf{P}(\text{Ci Champ})$
   $= 1 - (0.6)^2 * (0.5)^2 - (0.4)^2 * (0.3)^2 = 0.8956$

   (b) i. $\mathbf{P}(\text{Bo Challenger}|\text{2nd Rnd Req}) = \frac{(0.6)^2}{0.52} = \frac{0.36}{0.52} = \frac{9}{13}$
   ii. $\mathbf{P}(\text{Al Champ}|\text{2nd Rnd Req})$
   $= \frac{(0.6)^2*(0.5)^2+(0.6)^2*(0.5)+(0.4)^2*(0.3)*(0.7)+(0.4)^2*(0.7)}{0.52}$
   $= \frac{0.4156}{0.52} = 0.7992$

   (c) $\mathbf{P}((\text{Bo Challenger})|\{(\text{2nd Rnd Req}) \cap (\text{One Game})\}) = \frac{(0.6)^2*(0.5)}{(0.6)^2*(0.5)+(0.4)^2*(0.7)}$
   $= \frac{(0.6)^2(0.5)}{0.2920} = 0.6164$

3. (a) No. A and B are not independent. To see this, note that $A \subset B$, hence $\mathbf{P}(A \cap B) = \mathbf{P}(A)$. This is equal to $\mathbf{P}(A) \cdot \mathbf{P}(B)$ only when $\mathbf{P}(B) = 1$ or $\mathbf{P}(A) = 0$. But in our example, clearly $\mathbf{P}(B) < 1$ and $\mathbf{P}(A) > 0$. Hence $\mathbf{P}(A \cap B) \neq \mathbf{P}(A)\mathbf{P}(B)$, and thus A and B are not independent.

   (b) Yes. Conditioned on C, A will happen if and only if Imno meets 5 people during the second week. Hence $\mathbf{P}(A|C) = 1/5$.

   If Imno made 5 friends in the first week, she is certain to make more than 5 friends in total. Hence $\mathbf{P}(B|C) = 1$.

   If A happens, B will also happen, so clearly $\mathbf{P}(A \cap B|C) = \mathbf{P}(A|C) = \mathbf{P}(A|C) \cdot \mathbf{P}(B|C)$, therefore A and B are conditionally independent. Note that A and B were not independent prior to the conditioning.

   (c) No. We found in part (b) that $\mathbf{P}(A|C) = 1/5$, whereas $\mathbf{P}(A) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$. Hence A and C are not independent. (Note: $\mathbf{P}(A|C) = \frac{\mathbf{P}(A \cap C)}{\mathbf{P}(C)}$ by definition, and independence implies that $\mathbf{P}(A|C) = \frac{\mathbf{P}(A) \cdot \mathbf{P}(C)}{\mathbf{P}(C)} = \mathbf{P}(A)$. Hence $\mathbf{P}(A|C) = \mathbf{P}(A)$ is a necessary and sufficient condition for checking indepence, as long as $\mathbf{P}(C) > 0$.) $\mathbf{P}(B|C) = 1$, as we found above, but clearly $\mathbf{P}(B) < 1$, hence B and C are not independent.

   (d) Let F_i where $(i = 1, ..., 5)$ denote the event that in the first week i friends were made. Similarly let S_i denote the event that in the second week i friends were made. Let T_j where $(j = 2, ..., 10)$ denote the event that the total number of friends made in the two weeks is j.

$$\begin{aligned}
\mathbf{P}(\text{“2 in first''}|\text{“6 total''}) &\stackrel{def}{=} \mathbf{P}(F_2|T_6) \\
&= \frac{\mathbf{P}(T_6|F_2)\cdot\mathbf{P}(F_2)}{\mathbf{P}(T_6)} \\
&= \frac{\mathbf{P}(S_4\cdot)\mathbf{P}(F_2)}{\sum_{i=1}^{5}\mathbf{P}(F_i\cap S_{(6-i)})} \\
&= \frac{\frac{1}{5}\cdot\frac{1}{5}}{5\cdot\frac{1}{5}\frac{1}{5}} = \frac{1}{5},
\end{aligned}$$

where the second equality uses Bayes' Rule, and the third equality uses the Total Probability Theorem, and the last equality uses the fact that the numbers of friends made in each week are independent.

Similarly, $\mathbf{P}(F_3|T_6) = 1/5$.

(e)

$$\begin{aligned}
\mathbf{P}(F_2\cup S_2|T_6) &= \mathbf{P}(F_2|T_6) + \mathbf{P}(S_2|T_6) - \mathbf{P}(F_2\cap S_2|T_6) \\
&= \frac{1}{5} + \frac{1}{5} - 0 = \frac{2}{5}.
\end{aligned}$$

In the first equality we used the fact that conditional probabilities satisfy all the probability axioms. $\mathbf{P}(F_2|T_6)$ was found above to be 1/5. Since weeks are identically distributed, $\mathbf{P}(S_2|T_6)$ is also 1/5. The second equality follows.

$$\begin{aligned}
\mathbf{P}(F_3\cup S_3|T_6) &= \mathbf{P}(F_3|T_6) + \mathbf{P}(S_3|T_6) - \mathbf{P}(F_3\cap S_3|T_6) \\
&= \frac{1}{5} + \frac{1}{5} - \mathbf{P}(S_3|F_3\cap T_6)\cdot\mathbf{P}(F_3|T_6) \\
&= \frac{1}{5} + \frac{1}{5} - 1\cdot\frac{1}{5} = \frac{1}{5}.
\end{aligned}$$

Note that this result makes sense, since there is only one way to meet 6 people by meeting three in at least one week, whereas there are two ways of meeting 6 people by meeting two in at least one. The above probability is one half of the probability of the latter event, which is 2/5.