Tutorial 6 Solutions
Week of March 14, 2005

1. (a) To find the PDF $f_R(r)$ by convolution (assuming that X and Y are independent), first flip one of the PDF's. We'll flip $f_X(x)$. Then, to get $f_R(r)$ (i.e. the PDF of r at a specific point r), move the flipped graph so that its origin is at the point r, multiply the flipped graph with $f_Y(y)$, and integrate to get the result. Algebraically, the convolution is

$$f_R(r) = \int_{-\infty}^{\infty} f_Y(y) f_X(r-y)\, dy \ .$$

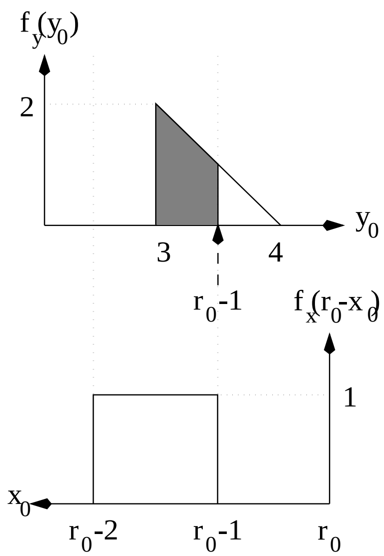


Fig 7.5a

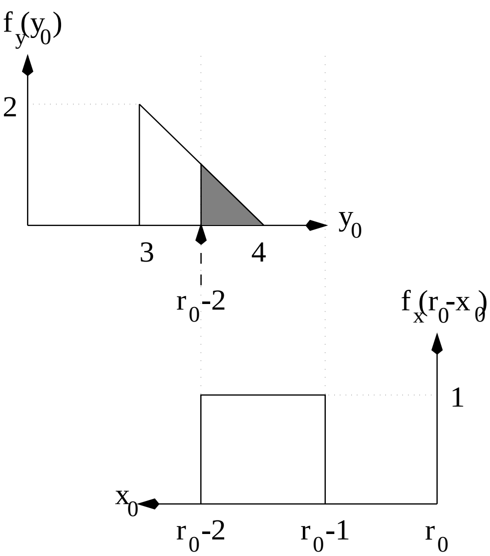


Fig 7.5b

In performing the convolution, we notice that

$$f_R(r_0) = 0 \ \text{ for } r_0 < 4 \text{ and } r_0 > 6$$

For $4 \le r_0 < 5$, $f_R(r_0)$ is the area of the shaded region in Figure 7.5a, which is

$$\begin{aligned} 1 - (\text{area of the unshaded triangle}) &= 1 - \frac{1}{2}(\text{base})(\text{height}) \\ &= 1 - \frac{1}{2}(4-(r_0-1))[2(4-(r_0-1))] \\ &= 1-(5-r_0)^2. \end{aligned}$$

(Note that the height of the triangle is two times the base.) For $5 \le r_0 \le 6$, $f_R(r_0)$ is the area of the shaded region in Figure 7.5b, which is

$$\begin{aligned} \frac{1}{2}(\text{base})(\text{height}) &= \frac{1}{2}(4-(r_0-2))[2(4-(r_0-2))] \\ &= (6-r_0)^2. \end{aligned}$$

Thus, the PDF of R is

$$\boxed{f_R(r) = \begin{cases} 1-(5-r)^2 & \text{for } 4 \le r < 5, \\ (6-r)^2 & \text{for } 5 \le r \le 6, \\ 0 & \text{otherwise.} \end{cases}}$$

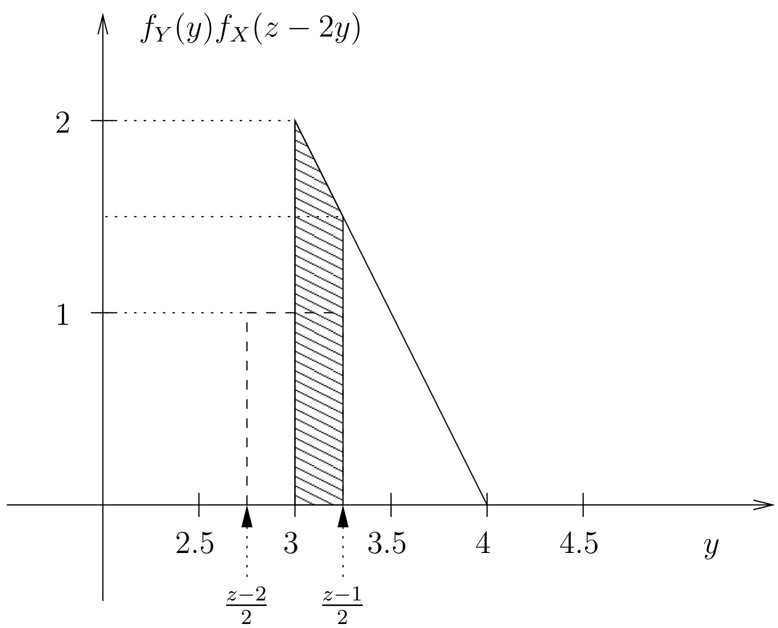
(b) First note that the PDF of Y is

$$f_Y(y) = \begin{cases} -2y+8 & \text{for } 3 \le y \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Also note that

$$f_X(z-2y) = \begin{cases} 1 & \text{for } 1 \le z-2y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

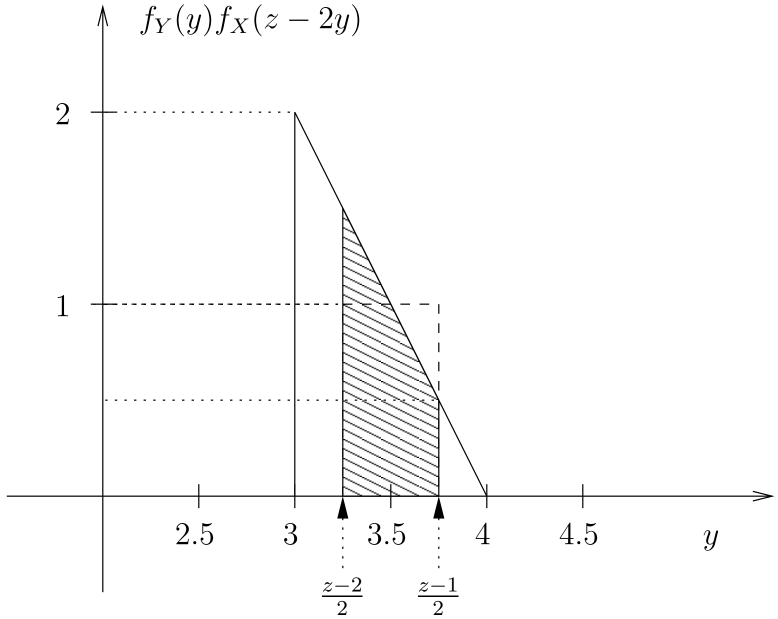
There are three cases to consider. The following figure illustrates the case $3 \le (z-1)/2 < 3.5$.



From this we get

$$\begin{aligned} f_Z(z) &= \text{(area of shaded region)} \\ &= 1 - \text{(area of unshaded triangle)} \\ &= 1-(4-(z-1)/2)(-2(z-1)/2+8)/2 \\ &- 1-(9-z)^2/4. \end{aligned}$$

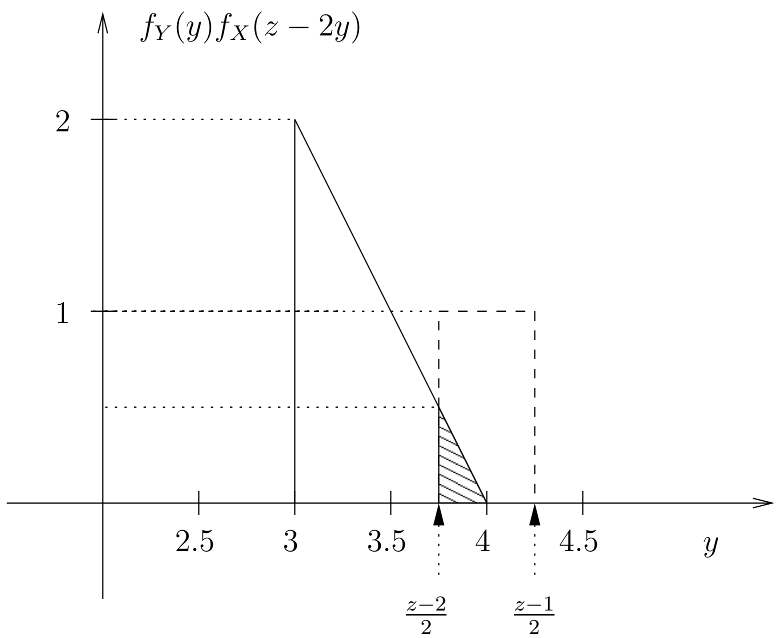
The following figure illustrates the case $3.5 \le (z-1)/2 < 4$.



From this we get

$$\begin{aligned} f_Z(z) &= \text{(area of shaded region)} \\ &= \text{(area of shaded rectangle with height } -2(z-1)/2+8) + \\ &\quad \text{(area of shaded triangle on top)} \\ &= (-2(z-1)/2+8)(1/2) + (-2(z-2)/2+8-(-2(z-1)/2+8))(1/2)/2 \\ &= (9-z)/2 + 1/4. \end{aligned}$$

The following figure illustrates the case $3.5 \leq (z-2)/2 \leq 4$.



From this we get

$$\begin{aligned} f_Z(z) &= \text{(area of shaded triangle)} \\ &= (4-(z-2)/2)(-2(z-2)/2+8)/2 \\ &= (10-z)^2/4. \end{aligned}$$

Putting everything together, we get that the derived PDF of Z is

$$f_Z(z) = \begin{cases} 1-(9-z)^2/4 & \text{for } 7 \le z < 8, \\ (9-z)/2+1/4 & \text{for } 8 \le z < 9, \\ (10-z)^2/4 & \text{for } 9 \le z \le 10, \\ 0 & \text{otherwise.} \end{cases}$$

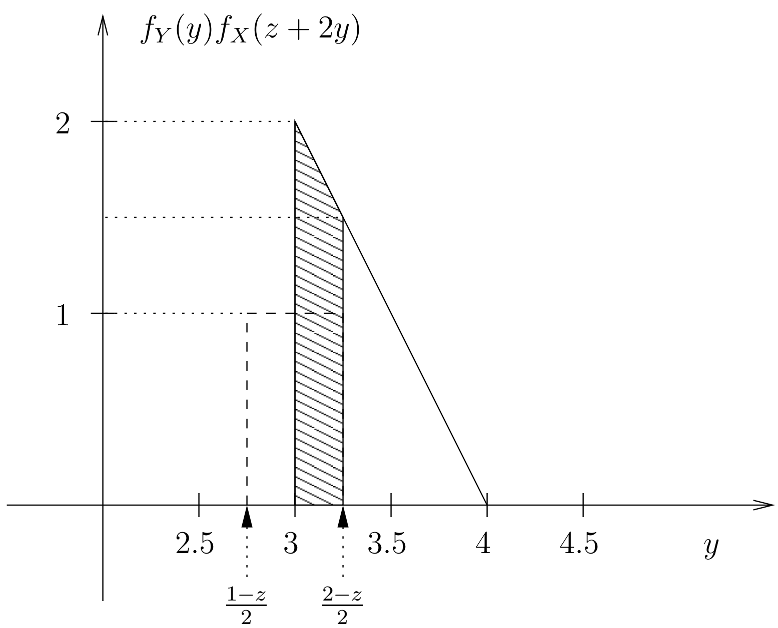
(c) First note that the PDF of Y is

$$f_Y(y) = \begin{cases} -2y+8 & \text{for } 3 \le y \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Also note that

$$f_X(z+2y) = \begin{cases} 1 & \text{for } 1 \le z+2y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

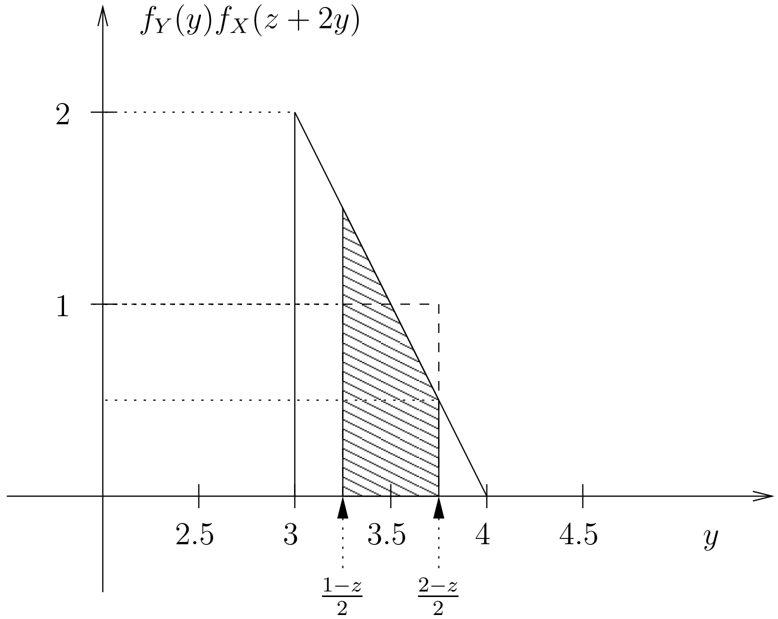
There are three cases to consider. The following figure illustrates the case $3 \le (2-z)/2 < 3.5$.



From this we get

$$\begin{aligned} f_Z(z) &= \text{(area of shaded region)} \\ &= 1-\text{(area of unshaded triangle)} \\ &= 1-(4-(2-z)/2)(-2(2-z)/2+8)/2 \\ &- 1-(6+z)^2/4. \end{aligned}$$

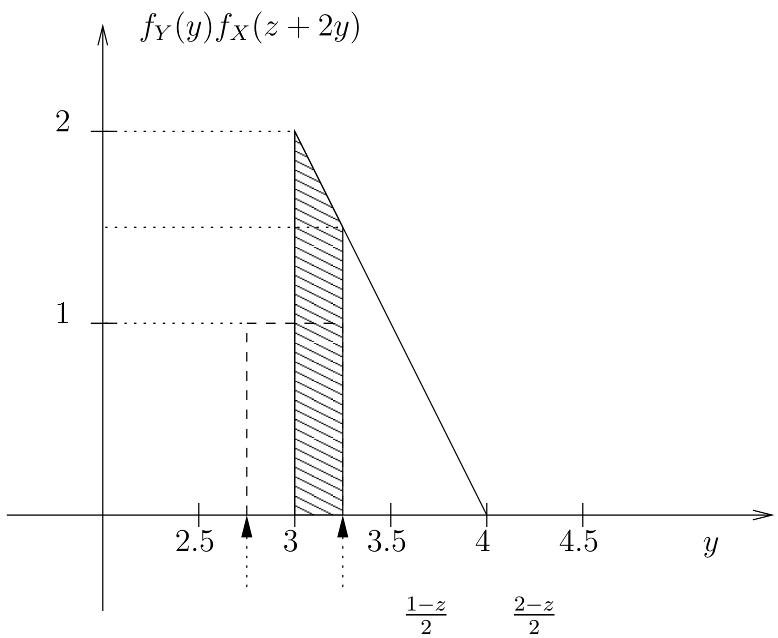
The following figure illustrates the case $3.5 \le (2-z)/2 < 4$.



From this we get

$$\begin{aligned}
f_Z(z) &= \text{(area of shaded region)} \\
&= \text{(area of shaded rectangle with height } -2(2-z)/2+8) + \\
&\quad \text{(area of shaded triangle on top)} \\
&= (-2(2-z)/2+8)(1/2) + (-2(1-z)/2+8-(-2(2-z)/2+8))(1/2)/2 \\
&= (6+z)/2+1/4.
\end{aligned}$$

The following figure illustrates the case $3.5 \le (1-z)/2 \le 4$.



From this we get

$$\begin{aligned} f_Z(z) &= \text{(area of shaded triangle)} \\ &= (4-(1-z)/2)(-2(1-z)/2+8)/2 \\ &= (7+z)^2/4. \end{aligned}$$

Putting everything together, we get that the derived PDF of Z is

$$f_Z(z) = \begin{cases} (7+z)^2/4, & \text{if } -7 \le z \le -6, \\ (6+z)/2+1/4, & \text{if } -6 < z \le -5, \\ 1-(6+z)^2/4, & \text{if } -5 < z \le -4, \\ 0, & \text{otherwise.} \end{cases}$$

2. (a) The definition of the transform is

$$M_Z(s) = \mathbf{E}[e^{sZ}]$$

Therefore, we know the following must be true:

$$M_Z(0) = \mathbf{E}[e^{0Z}] = \mathbf{E}[1] = 1.$$

So in our case

$$M_Z(0) = \frac{a}{8} = 1$$

and

$$a = 8.$$

(b) We approach this problem by first finding the PDF of Z using partial fraction expansion:

$$\begin{aligned} M_Z(s) &= \frac{8-3s}{s^2-6s+8} = \frac{A}{s-4} + \frac{B}{s-2} \\ A &= (s-4)M_Z(s)\Big|_{s=4} = \frac{8-3s}{s-2}\Big|_{s=4} = -2 \\ B &= (s-2)M_Z(s)\Big|_{s=2} = \frac{8-3s}{s-4}\Big|_{s=2} = -1. \end{aligned}$$

Thus,

$$M_Z(s) = \frac{-2}{s-4} + \frac{-1}{s-2} = \frac{1}{2}\left(\frac{4}{4-s} + \frac{2}{2-s}\right)$$

and

$$f_Z(z) = \begin{cases} \frac{1}{2}\left(4e^{-4z} + 2e^{-2z}\right) & \text{for } z \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

From this we get

$$\mathbf{P}(Z \ge 0.5) = \int_{0.5}^{\infty} \tfrac{1}{2}(4e^{-4z} + 2e^{-2z})dz = \boxed{\tfrac{e^{-2}}{2} + \tfrac{e^{-1}}{2}}.$$

(c) $\mathbf{E}[Z] = \int_0^{\infty} \frac{z}{2}(4e^{-4z} + 2e^{-2z})dz = \frac{1}{2}(\int_0^{\infty} 4ze^{-4z}dz + \int_0^{\infty} 2ze^{-2z}dz) = \frac{1}{2}(\frac{1}{4} + \frac{1}{2}) = \boxed{\frac{3}{8}}$

(d) $\mathbf{E}[Z] = \frac{d}{ds}M_Z(s)\Big|_{s=0} = \frac{d}{ds}(\frac{2}{4-s} + \frac{1}{2-s})\Big|_{s=0} = \frac{2}{(4-s)^2} + \frac{1}{(2-s)^2}\Big|_{s=0} = \boxed{\frac{3}{8}}$

(e) $\text{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2$
$\mathbf{E}[Z^2] = \int_0^\infty \frac{z^2}{2}(4e^{-4z} + 2e^{-2z})dz = \frac{1}{2}(\int_0^\infty 4z^2e^{-4z}dz + \int_0^\infty 2z^2e^{-2z}dz) = \frac{1}{2}(\frac{2}{4^2} + \frac{2}{2^2}) = \frac{5}{16}$
$\text{var}(Z) = \frac{5}{16} - (\frac{3}{8})^2 = \boxed{\frac{11}{64}}$

(f) $\mathbf{E}[Z^2] = \frac{d^2}{ds^2}M_Z(s)\Big|_{s=0} = \frac{d^2}{ds^2}(\frac{2}{4-s} + \frac{1}{2-s})\Big|_{s=0} = \frac{4}{(4-s)^3} + \frac{2}{(2-s)^3}\Big|_{s=0} = \frac{5}{16}$
$\text{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2 = \frac{5}{16} - (\frac{3}{8})^2 = \boxed{\frac{11}{64}}$