MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Problem Set 3G: Solutions
Due: February 23, 2005

G1. We know that the PMF for a binomial random variable is:

$$\mathbf{P}(X = k) \;=\; \binom{n}{k} p^k (1-p)^{n-k}$$

To find the value of p that maximizes $\mathbf{P}(X = k)$, we use elementary calculus to determine which value of p maximizes the a priori probability for any experimental value of k.

$$\begin{aligned}
\frac{d}{dp}[\mathbf{P}(X = k)] = \binom{n}{k} \left[kp^{k-1}(1-p)^{n-k} - p^k(n-k)(1-p)^{n-k-1}\right] &= 0 \\
k(1-p) - p(n-k) &= 0\,* \\
k - pk - pn + pk &= 0 \\
p &= \boxed{\text{k/n}}
\end{aligned}$$

For instance, if we observed 20 heads in a series of 45 coin tosses, then according to our maximum likelihood estimator, the best estimate for p (the probability of a head on any toss) would be $p = \frac{20}{45}$

*note, we divided through by the terms p^{k-1} and $(1-p)^{n-k-1}$ in step 2

G2. Notice that only the relative distance between the fly and the spider matters here, and not the absolute positions of the fly and the spider.
Denote:

- A_d the event that initially the spider and the fly are d units apart.
- B_d the event that after one second the spider and the fly are d units apart.

Our approach will be to first apply the (conditional version of the) total expectation theorem to compute $\mathbf{E}(T \mid A_1)$, then use the result to compute $\mathbf{E}(T \mid A_2)$, and similarly compute sequentially $\mathbf{E}(T \mid A_d)$ for all relevant values of d. We will then apply the (unconditional version of the) total expectation theorem to compute $\mathbf{E}(T)$.

We have

$$A_d = A_d B_d + A_d B_{d-1} + A_d B_{d-2}, \qquad \text{if } d > 1.$$

This is because if the spider and the fly are at a distance $d > 1$ apart, then one second later their distance will be d (if the fly moved away from the spider) or $d - 1$ (if the fly did not move) or $d - 2$ (if the fly moved towards the spider). We also have, for the case where the spider and the fly start one unit apart,

$$A_1 = A_1 B_1 + A_1 B_0.$$

Using the total expectation theorem, we obtain

$$\begin{array}{rcll} \mathbf{E}(T \mid A_d) & = & \mathbf{P}(A_dB_d)\mathbf{E}(T \mid A_dB_d) & \\ & & +\mathbf{P}(A_dB_{d-1})\mathbf{E}(T \mid A_dB_{d-1}) & \quad \text{if } d > 1, \\ & & +\mathbf{P}(A_dB_{d-2})\mathbf{E}(T \mid A_dB_{d-2}) & \end{array}$$

while for the case $d = 1$,

$$\mathbf{E}(T \mid A_1) = \mathbf{P}(A_1B_1)\mathbf{E}(T \mid A_1B_1) + \mathbf{P}(A_1B_0)\mathbf{E}(T \mid A_1B_0).$$

It can be seen based on the problem data that

$$\mathbf{P}(A_1B_1) = 2p, \qquad \mathbf{P}(A_1B_0) = 1 - 2p,$$
$$\mathbf{E}(T \mid A_1B_1) = 1 + \mathbf{E}(T \mid A_1), \qquad \mathbf{E}(T \mid A_1B_0) = 1,$$

so by applying the theorem with $d = 1$, we obtain

$$\mathbf{E}(T \mid A_1) = 2p\,(1 + \mathbf{E}(T \mid A_1)) + (1 - 2p),$$

or

$$\mathbf{E}(T \mid A_1) = \frac{1}{1 - 2p}.$$

By applying the theorem with $d = 2$, we obtain

$$\mathbf{E}(T \mid A_2) = p\mathbf{E}(T \mid A_2B_2) + (1 - 2p)\mathbf{E}(T \mid A_2B_1) + p\mathbf{E}(T \mid A_2B_0).$$

We have

$$\begin{array}{rcl} \mathbf{E}(T \mid A_2B_0) & = & 1, \\ \mathbf{E}(T \mid A_2B_1) & = & 1 + \mathbf{E}(T \mid A_1), \\ \mathbf{E}(T \mid A_2B_2) & = & 1 + \mathbf{E}(T \mid A_2), \end{array}$$

so by substituting these relations in the expression for $\mathbf{E}(T \mid A_2)$, we obtain

$$\begin{array}{rcl} \mathbf{E}(T \mid A_2) & = & p\,(1 + \mathbf{E}(T \mid A_2)) + (1 - 2p)\,(1 + \mathbf{E}(T \mid A_1)) + p \\ & = & p\,(1 + \mathbf{E}(T \mid A_2)) + (1 - 2p) \cdot \left(1 + \dfrac{1}{1 - 2p}\right) + p. \end{array}$$

This equation yields after some calculation

$$\mathbf{E}(T \mid A_2) = \frac{2}{1 - p}.$$

Similarly, we obtain for $d > 2$,

$$\begin{array}{rcl} \mathbf{E}(T \mid A_d) & = & p\,(1 + \mathbf{E}(T \mid A_d)) \\ & & +(1 - 2p)\,(1 + \mathbf{E}(T \mid A_{d-1})) \\ & & +p\,(1 + \mathbf{E}(T \mid A_{d-2}))\,, \end{array}$$

so $\mathbf{E}(T \mid A_d)$ can be generated recursively for any initial distance d, using as initial conditions the values of $\mathbf{E}(T \mid A_1)$ and $\mathbf{E}(T \mid A_2)$ obtained earlier.

Finally, the expected value of T can then be obtained using the given PMF for the initial distance d and the total expectation theorem:

$$\mathbf{E}(T) = \sum_{d_0} p_d(d_0)\mathbf{E}(T \mid A_{d_0}).$$