Recitation 13 Solutions
March 31, 2005

1. Let X be a normal random variable with mean 1 and variance σ^2. Let Y be equal to ± 1 with equal probability. Let $Z = XY$. It can be checked that X and Z are uncorrelated but they are obviously not independent.

2. Let A_t (respectively, B_t) be a Bernoulli random variable which is equal to 1 if and only if the tth toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_t B_t] = 0$ and $\mathbf{E}[A_t B_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = p_1 p_2$ for $s \neq t$. We have

$$\mathbf{E}[X_1 X_2] = \mathbf{E}\big[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\big] = n\mathbf{E}\big[A_1(B_1 + \cdots + B_n)\big] = n(n-1)p_1 p_2,$$

and

$$\text{cov}(X_1, X_2) = \mathbf{E}[X_1 X_2] - \mathbf{E}[X_1]E[X_2] = n(n-1)p_1 p_2 - np_1 np_2 = -np_1 p_2.$$

3. (a) First note that $\hat{X}$ should be a r.v., not a number. In particular, we are to minimize over all r.v.'s $\hat{X}$ that can be expressed as functions of Y. From lecture, $\hat{X} = \mathbf{E}[X|Y]$.

   Now, take conditional expectations, to get $Y = \mathbf{E}[Y|Y] = \mathbf{E}[X|Y] + \mathbf{E}[W|Y]$. Since there is complete symmetry between X and W, we also have $\mathbf{E}[X|Y] = \mathbf{E}[W|Y]$, which finally yields $\mathbf{E}[X|Y] = Y/2$.

   (b) In the dependent case, we cannot simply conclude that the distribution $f_{X,W}(x,w)$ is symmetric in its two argument (i.e., $f_{X,W}(x,w) = f_{X,W}(w,x)$), even though the marginals $f_X(x), f_W(w)$ are the same.
   Since $f_{X,W}(x,w)$ is not symmetric, $\mathbf{E}[X|Y] \neq \mathbf{E}[W|Y]$ in general.
   So in this case, one cannot really solve the problem with the available information, we really need the joint distribution in order to compute the conditional expectations.
   The solution given in the independent case still works, though, for any symmetric distribution.

4. The solution can be found on page 264 in the textbook.