6.041 Final Exam Solutions
May 17, 2005

Problem 1: (26 points)

(a) (6 points) We have

$$\mathrm{var}(Y_n) = \frac{\mathrm{var}\left(\sum_{i=1}^{n} X_i\right)}{n} = \frac{n\sigma^2}{n} = \sigma^2.$$

Hence, by the Chebyshev inequality,

$$\mathbf{P}(|Y_n - \mathbf{E}[Y_n]| \geq \epsilon) \leq \frac{\mathrm{var}(Y_n)}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}.$$

(b) (7 points) Since Y_n is a linear function of independent normal random variables, it is normal. Hence

$$\mathbf{P}(|Y_n - \mathbf{E}[Y_n]| \geq \epsilon) = \mathbf{P}\left(\frac{|Y_n - \mathbf{E}[Y_n]|}{\sqrt{\mathrm{var}(Y_n)}} \geq \frac{\epsilon}{\sigma}\right) = 2\left(1 - \Phi\left(\frac{\epsilon}{\sigma}\right)\right).$$

(c) (6 points)

ϵ/σ	$\mathbf{P}(\lvert Y_n - \mathbf{E}[Y_n]\rvert \geq \epsilon)$	Chebyshev bound
0.5	$2(1-\Phi(0.5)) = 2(1-0.6915) = 0.6170$	4.0
1.0	$2(1-\Phi(1.0)) = 2(1-0.8413) = 0.3174$	1.0
2.0	$2(1-\Phi(2.0)) = 2(1-0.9772) = 0.0456$	0.25

(d) (7 points) We have

$$Y_n = \frac{\sqrt{k}Y_k + \sum_{i=k+1}^{n} X_i}{\sqrt{n}}.$$

Therefore, the least squares estimate of Y_n given $Y_k = y$ is given by

$$\mathbf{E}[Y_n | Y_k = y] = \mathbf{E}\left[\left.\frac{\sqrt{k}Y_k + \sum_{i=k+1}^{n} X_i}{\sqrt{n}}\right| Y_k = y\right] = \frac{\sqrt{k}y + (n-k)\mu}{\sqrt{n}},$$

which, being a linear function of y, must be equal to the linear least squares estimate. Alternatively, we compute

$$\mathrm{cov}(Y_n, Y_k) = \mathrm{cov}\left(\sqrt{\frac{k}{n}}Y_k, Y_k\right) + \mathrm{cov}\left(\frac{\sum_{i=k+1}^{n} X_i}{\sqrt{n}}, Y_k\right) = \mathrm{cov}\left(\sqrt{\frac{k}{n}}Y_k, Y_k\right).$$

Now,

$$\mathrm{cov}\left(\sqrt{\frac{k}{n}}Y_k, Y_k\right) = \mathbf{E}\left[\left(\sqrt{\frac{k}{n}}Y_k - \mathbf{E}\left[\sqrt{\frac{k}{n}}Y_k\right]\right)(Y_k - \mathbf{E}[Y_k])\right] = \sqrt{\frac{k}{n}}\mathrm{var}(Y_k).$$

Hence the linear least squares estimate of Y_n given $Y_k = y$ is given by

$$\mathbf{E}[Y_n] + \frac{\mathrm{cov}(Y_n, Y_k)}{\mathrm{var}(Y_k)}(y - \mathbf{E}[Y_k]) = \sqrt{n}\mu + \sqrt{\frac{k}{n}}(y - \sqrt{k}\mu) = \frac{\sqrt{k}y + (n-k)\mu}{\sqrt{n}}.$$

The mean square error of the estimate is

$$\mathbf{E}\left[\left(\frac{\sqrt{k}Y_k+\sum_{i=k+1}^{n}X_i}{\sqrt{n}}-\frac{\sqrt{k}y+(n-k)\mu}{\sqrt{n}}\right)^2\middle| Y_k=y\right]=\frac{\sum_{i=k+1}^{n}\mathrm{var}(X_i)}{n}=\left(1-\frac{k}{n}\right)\sigma^2.$$

Problem 2: (36 points)

(a) (5 points) Vehicles arrive according to a Poisson process with rate $\lambda_1+\lambda_2$. Owing to the memorylessness of the Poisson process, the time until the first vehicle after 7:00 a.m. is distributed as an exponential random variable with parameter $\lambda_1+\lambda_2$. Therefore, the average time before they see the first vehicle is $1/(\lambda_1+\lambda_2)$.

(b) (5 points) By the properties of merged Poisson processes, we have

$$\mathbf{P}(\text{first vehicle is a taxi})=\frac{\lambda_1}{\lambda_1+\lambda_2},$$

and

$$\mathbf{P}(\text{first vehicle is a bus})=\frac{\lambda_2}{\lambda_1+\lambda_2}.$$

(c) (7 points) We have $X=W+D$, where W is the time until the arrival of the first vehicle, and D is the travel time. Now,

$$M_W(s)=\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-s},$$

since, as established in part (a), W is an exponential random variable with parameter $\lambda_1+\lambda_2$. As for D, we have

$$\begin{aligned}M_D(s)&=\mathbf{E}\left[e^{sD}\right]\\&=\mathbf{E}\left[e^{sD}\middle|\text{first vehicle is a taxi}\right]\frac{\lambda_1}{\lambda_1+\lambda_2}+\mathbf{E}\left[e^{sD}\middle|\text{first vehicle is a bus}\right]\frac{\lambda_2}{\lambda_1+\lambda_2}\\&=M_{D_T}(s)\frac{\lambda_1}{\lambda_1+\lambda_2}+M_{D_B}(s)\frac{\lambda_2}{\lambda_1+\lambda_2}=\frac{\mu_1}{\mu_1-s}\frac{\lambda_1}{\lambda_1+\lambda_2}+\frac{\mu_2}{\mu_2-s}\frac{\lambda_2}{\lambda_1+\lambda_2}.\end{aligned}$$

Therefore,

$$M_X(s)=M_W(s)M_D(s)=\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-s}\left(\frac{\mu_1}{\mu_1-s}\frac{\lambda_1}{\lambda_1+\lambda_2}+\frac{\mu_2}{\mu_2-s}\frac{\lambda_2}{\lambda_1+\lambda_2}\right).$$

(d) (7 points) We have $Y=\max(D_T,D_B)$. We write $Y=Y_1+Y_2$, where $Y_1=\min(D_T,D_B)$ denotes the time till the arrival of the first vehicle, and $Y_2=Y-Y_1$ denotes the time from the arrival of the first vehicle till the arrival of the second vehicle. First, we have

$$\mathbf{E}[Y_1]=\frac{1}{\mu_1+\mu_2}.$$

Then

$$\begin{aligned}\mathbf{E}[Y_2]&=\mathbf{E}[Y_2|D_T<D_B]\mathbf{P}(D_T<D_B)+\mathbf{E}[Y_2|D_T\geq D_B]\mathbf{P}(D_T\geq D_B)\\&=\frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2}+\frac{1}{\mu_1}\frac{\mu_2}{\mu_1+\mu_2}.\end{aligned}$$

So

$$\mathbf{E}[Y] = \frac{1}{\mu_1 + \mu_2}\left(1 + \frac{\mu_1}{\mu_2} + \frac{\mu_2}{\mu_1}\right).$$

(e) (5 points) Fast buses arrive according to a Poisson process with rate $p\lambda_2$. Hence the average number of fast buses they will see is $lp\lambda_2$.

(f) (7 points) The probability that exactly i slow buses arrive before the arrival of the kth fast bus is the probability that the k fast bus is the $(k+i)$th bus, which, using the Pascal PMF, is given by

$$\binom{k+i-1}{k-1} p^k (1-p)^i.$$

The probability that they see k fast buses before they see k slow buses is the probability that strictly less than k slow buses arrive before the arrival of the kth fast bus, which is therefore given by

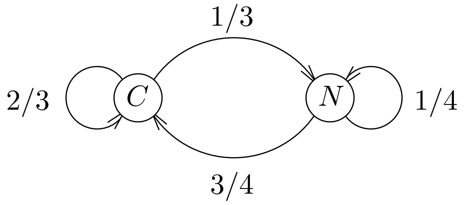
$$\sum_{i=0}^{k-1} \binom{k+i-1}{k-1} p^k (1-p)^i.$$

Alternatively, the probability of seeing k fast buses before k slow buses is equivalent to the probability of seeing k or more fast buses in the first $2k-1$ buses, which is given by

$$\sum_{i=k}^{2k-1} \binom{2k-1}{i} p^i (1-p)^{2k-1-i}.$$

Problem 3: (35 points)

(a) (7 points) We model the problem with the following two-state Markov chain, where C represents the state where cheese is found, and N represents the state where no cheese is found.



Therefore, the steady-state probabilities satisfy

$$\frac{1}{3}\pi_C = \frac{3}{4}\pi_N$$

and

$$\pi_C + \pi_N = 1,$$

whence we conclude that $\pi_N = 4/13$ and $\pi_C = 9/13$. Thus, over 1000 turns, we expect that the mouse eats a piece of cheese in approximately 9/13 of them.

(b) (7 points) We wish to know the mean first passage time t_N to reach state C from state N. We have

$$t_N = 1 + \frac{1}{4} t_N,$$

which implies that $t_N = 4/3$.

(c) (7 points) We wish to know the mean first passage time to reach state C from state N followed by $(n-1)$ recurrence times of state C. The mean recurrence time t_C^* of state C is given by

$$t_C^* = 1 + \frac{1}{3} t_N = 1 + \frac{4}{9} = \frac{13}{9}.$$

Hence the expected number of turns until the mouse eats n pieces of cheese is $4/3+(n-1)13/9$.

(d) (7 points) Let Y_{100} be the number of turns before the mouse eats 100 pieces of cheese. Then, $Y_{100} = \sum_{i=1}^{100} R_i$, where R_i denotes the number of turns from eating the $(i-1)$th piece of cheese till eating the ith piece of cheese. The random variable R_i is equal to 1 with probability $2/3$ (transition from C to C) and equal to $1+S$ with probability $1/3$ (transition from C to T), where S is a geometric random variable with parameter $3/4$. We have already established that $\mathbf{E}[R_i] = 13/9$. Therefore,

$$\begin{aligned}\mathbf{E}[R_i^2] &= 1 \cdot \frac{2}{3} + \mathbf{E}[(1+S)^2] \cdot \frac{1}{3} = \frac{2}{3} + \frac{1}{3}(1 + 2\mathbf{E}[S] + \mathbf{E}[S^2]) \\ &= \frac{2}{3} + \frac{1}{3}(1 + 2\mathbf{E}[S] + \mathrm{var}(S) + (\mathbf{E}[S])^2) = \frac{2}{3} + \frac{1}{3}\left(1 + \frac{8}{3} + \frac{1/4}{(3/4)^2} + \frac{16}{9}\right) = \frac{71}{27}.\end{aligned}$$

Hence

$$\mathrm{var}(R_i) = \frac{71}{27} - \left(\frac{13}{9}\right)^2 = \frac{44}{81}.$$

Since it is clear that $R_1, R_2, \ldots, R_{100}$ is an i.i.d. sequence, it follows by the Central Limit Theorem that

$$\begin{aligned}\mathbf{P}(Y_{100} > 152) &= \mathbf{P}\left(\frac{Y_{100} - \mathbf{E}[Y_{100}]}{\sqrt{\mathrm{var}(Y_{100})}} > \frac{152 - 100 \cdot 13/9}{10 \cdot \sqrt{44/81}}\right) = \mathbf{P}\left(\frac{Y_{100} - \mathbf{E}[Y_{100}]}{\sqrt{\mathrm{var}(Y_{100})}} > \sqrt{\frac{289}{275}}\right) \\ &\approx 1 - \Phi(1.03) = 1 - 0.8485 = 0.1515.\end{aligned}$$

(e) (7 points) We now model the problem with a four-state Markov chain. The states are $C1$, $C2$, $C3$, and N, where $C1$, $C2$, and $C3$ represent the states where cheese is found behind the first, second, and third doors, respectively, and N represents the state where no cheese is found. The transition probabilities are

$$\begin{aligned} p_{CiCj} &= \begin{cases} 1/3 & \text{if } j \neq i, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } i, j = 1, 2, 3, \\ p_{CiN} &= 1/3, \qquad \text{for all } i = 1, 2, 3, \\ p_{NCi} &= 1/4, \qquad \text{for all } i = 1, 2, 3, \end{aligned}$$

and

$$p_{NN} = 1/4.$$

There are seven three-transition paths from state $C1$ to state $C1$, which are

(i) $C1 \rightarrow C2 \rightarrow C3 \rightarrow C1$,
(ii) $C1 \rightarrow C2 \rightarrow N \rightarrow C1$,
(iii) $C1 \rightarrow C3 \rightarrow C2 \rightarrow C1$,
(iv) $C1 \rightarrow C3 \rightarrow N \rightarrow C1$,
(v) $C1 \rightarrow N \rightarrow C2 \rightarrow C1$,
(vi) $C1 \rightarrow N \rightarrow C3 \rightarrow C1$, and
(vii) $C1 \rightarrow N \rightarrow N \rightarrow C1$.

Paths (i) and (iii) occur with probability 1/27, paths (ii), (iv), (v), and (vi) occur with probability 1/36, and path (vii) occurs with probability 1/48. Therefore, the probability of going from state $C1$ to state $C1$ in three transitions is

$$\frac{2}{27} + \frac{4}{36} + \frac{1}{48} = \frac{89}{432}.$$