**Recitation 17- Solutions**
**April 14, 2004**

1. (a) $T = \min(T_F, T_R)$, so $T$ is exponential with parameter $\lambda = \lambda_F + \lambda_R$. (The minimum of independent exponential random variables was discussed in a previous recitation.)

   (b) Now with the instantaneous repairs, the front tire flats are arrivalsin an arrival process. Since the interarrival times are exponential, the process is a Poisson process. Thus the number of times the front tire goes flat by time $t$ is a Poisson random variable with mean $\lambda_F t$. Similarly the number of times the rear tire goes flat by time $t$ is a Poisson random variable with mean $\lambda_R t$.

   (c)

$$\mathbf{P}(\text{one front flat in } [0,t] \mid \text{one flat total in } [0,t]) = \frac{\mathbf{P}(\text{one front flat})\mathbf{P}(\text{zero rear flats})}{\mathbf{P}(\text{one flat total})} = \frac{e^{-\lambda_F t}\lambda_F t \cdot e^{-\lambda_R t}}{e^{-(\lambda_F+\lambda_R)t}(\lambda_F+\lambda_R)t} = \frac{\lambda_F}{\lambda_F+\lambda_R}$$

   (d) Because $T_F$ and $T_R$ are independent, the joint PDF is just the product of the individual exponential PDFs:

$$f_{T_F,T_R}(t_F,t_R) = \lambda_F\lambda_R e^{-\lambda_F t_F} e^{-\lambda_R t_R} \quad \text{for } t_F,\ t_R \geq 0;\ 0 \text{ otherwise.}$$

   The probability that the front tire goes flat before the rear tire is

$$\mathbf{P}(T_F < T_R) = \int_0^\infty \int_{t_F}^\infty f_{T_F,T_R}(t_F,t_R)\,dt_R\,dt_F = \frac{\lambda_F}{\lambda_F+\lambda_R}.$$

   (e) This is an exponential with rate $\lambda_F + \lambda_R$, just in Part (a), i.e. the distribution of $T$ is independent of whether the first tire to go flat is the front or rear one. You can derive this by finding the conditional cumulative of $T$ conditional on $T_F < T_R$, and then differentiating.

   (f) This is just an application of the Total Probability Theorem. Let $T_S$ be the time untill the tire with stickers on goes flat. Then $f_{T_S}(t) = \frac{1}{2}\lambda_F e^{-\lambda_F t} + \frac{1}{2}\lambda_R e^{-\lambda_R t}$. Also the transform of $T_S$ is $M_{T_S}(s) = \frac{1}{2}M_1(s) + \frac{1}{2}M_2(s)$, where $M_1$ and $M_2$ are the transforms of $\lambda_F e^{-\lambda_F t}$ and $\lambda_R e^{-\lambda_R t}$, respectively.

2. (a) Since we are looking for the number of "trials" up to and including the first "success," $N$ is a geometric random variable with parameter $p$.

$$p_N(n) = (1-p)^{n-1}p, \quad n \geq 1.$$

   (b) The length of time spent driving to each intersection is exponentially distributed with parameter $\lambda$. Since the probability of Shem observing an accident at a given intersection is $p$, the distribution of the length of time in between accident reports is exponential but with parameter $p\lambda$ (think of Poisson splitting). Thus,
   $f_Q(q) = (p\lambda)e^{-qp\lambda}$, $q \geq 0$.

(c) Since the interarrival time of accidents is exponentially distributed with parameter $p\lambda$, the number of arrivals in a given amount of time $\tau$ is a Poisson random variable with parameter $p\lambda\tau$. Thus,
$\mathbf{P}(m \text{ arrivals in 2 hours}) = p_M(m) = \frac{e^{-2p\lambda}(2p\lambda)^m}{m!}, \quad m \geq 0.$

(d) We can view the radio calls to Shem and the accident reports as independent Poisson processes with arrival rates $\mu$ and $p\lambda$, respectively. When the two independent Poisson processes are joined, the resultant is a Poisson process with arrival rate $\mu + p\lambda$. Furthermore, the probability of an arrival from the radio calls is $\frac{\mu}{\mu+p\lambda}$. Since we are interested in the number of reported accidents between two radio calls, we can view this is a shifted Geometric random variable with parameter $\frac{\mu}{\mu+p\lambda}$. Thus,
$p_K(k) = (\frac{p\lambda}{\mu+p\lambda})^k(\frac{\mu}{\mu+p\lambda}), \quad k \geq 0.$

(e) i. If we begin to observe Shem's radio calls at some random instant in time, due to the memoryless property of Poisson interarrivals, the distribution until he recieves the next call will still be exponential with parameter $\mu$. Also, the time from the previous call until the point at which we begin to observe Shem is also an exponential distribution with parameter $\mu$. Thus, $W = X_1 + X_2$, where $X_1$ and $X_2$ have exponential distributions, i.e. $W$ is a second order Erlang PDF.
$f_W(w) = (\mu)^2 we^{-w\mu}$

ii. Since $X_1$ and $X_2$ are independent, the transform of $W$ is the product of the transforms of $X_1$ and $X_2$, where each has the transform $M_X(s) = \frac{\mu}{\mu-s}$. Thus,
$M_W(s) = (\frac{\mu}{\mu-s})^2.$

(f) The time $V$ that we are looking for consists of the the time between our observation of Shem and his next accident report, plus the time between his next accident report and when he receives his next radio call. These two times are independent of each other. If we call the former time $Y$, and the latter time $Z$, we notice that, due to the memoryless property, $Y$ is exponential with parameter $\lambda p$, and $Z$ is exponential with parameter $\mu$.

$$M_V(s) = M_Y(s)M_Z(s) = (\frac{\lambda p}{\lambda p - s})(\frac{\mu}{\mu - s}).$$