Recitation 12 Solutions
March 29, 2005

1. For both parts (a) and (b) we will make use of the formulas:

$$\begin{aligned} \mathbf{E}[X] &= \mathbf{E}[\mathbf{E}[X|Y]] \\ \text{var}(X) &= \mathbf{E}[\text{var}(X|Y)] + \text{var}(\mathbf{E}[X|Y]) \end{aligned}$$

Let X be the number of heads, and let Y be the result of the roll.

(a)

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y] = \mathbf{E}[Y/2] = \frac{7}{4}.$$

and similarly,

$$\text{var}(X) = \mathbf{E}[\text{var}(X|Y)] + \text{var}(\mathbf{E}[X|Y]) = \frac{7}{2}\text{var}(X) + \text{var}(Y/2) = \frac{77}{48}.$$

(b) For this part, let X_1 be the number of heads that correspond to the first die roll, and X_2 be the number of heads that correspond to the second die roll. Clearly $X = X_1 + X_2$ and X_1, X_2 are iid. Thus we have

$$\mathbf{E}[X] = \mathbf{E}[X_1 + X_2] = 2\mathbf{E}[X_1] = 2 \cdot \frac{7}{4} = \frac{7}{2}.$$

Similarly,

$$\text{var}(X) = \text{var}(X_1 + X_2) = 2\text{var}(X_1) = 2 \cdot \frac{77}{48} = \frac{77}{24}.$$

2. (a) For any particular crate, let X_i be the number of widgets in the ith box in that crate. Note that the X_i's have the same mean and variance as X. Then $T = X_1 + \cdots + X_N$ which is the sum of a random number of independent random variables. By the properies of such random variables, we have

$$\mathbf{E}[T] = \mathbf{E}[X]\mathbf{E}[N] = 10 \cdot 10 = 100,$$

and

$$\text{var}(T) = \text{var}(X)\mathbf{E}[N] + (\mathbf{E}[X])^2\text{var}(N) = 16 \cdot 10 + (10)^2 \cdot 16 = 1760.$$

(b) Let T_j be the number of boxes in the jth crate in the particular shipment. Then $W = T_1 + \cdots + T_K$ which is the sum of a random number of independent random variables with the same mean and variance as T. Therefore, by the properties of W, we have

$$\mathbf{E}[W] = \mathbf{E}[T]\mathbf{E}[K] = 100 \cdot 10 = 1000,$$

and

$$\text{var}(W) = \text{var}(T)\mathbf{E}[N] + (\mathbf{E}[T])^2\text{var}(N) = 1760 \cdot 10 + (100)^2 \cdot 16 = 177600.$$

3. (a) $\mathbf{P}(|X_1| \le \delta) \approx \alpha\delta$.
$\mathbf{P}(-\delta \le X_1 \le \delta) = \int_{-\delta}^{\delta} f_{X_1}(x)dx_1 = 2\delta \cdot f_{X_1}(0) = \delta \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}}$.
$\alpha = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}}$.

(b) $\mathbf{E}[X_1 N] = \mathbf{E}[X_1]\mathbf{E}[N] = \frac{3}{2} \cdot 2 = 3.$

(c) $\text{var}(X_1 N) = \mathbf{E}[X_1^2 N^2] - (\mathbf{E}[X_1 N])^2 = (4+4)3 - 3^2 = 15.$

(d)

$$\begin{aligned}
\mathbf{E}[X_1 + \cdots + X_N] &= \mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2]\mathbf{P}(N \geq 2) + \\
&\quad \mathbf{E}[X_1 + \cdots + X_N \mid N < 2]\mathbf{P}(N < 2). \\
3 &= \mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2](1-p) + \mathbf{E}[X_1](p). \\
\mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2] &= 3(3 - 2(2/3)) = 5.
\end{aligned}$$

(e) Let $Z = N + X_1 + \cdots + X_N$. Note that N and $X_1 + \cdots + X_N$ are NOT independent.

$M_Z(s) = \mathbf{E}[\mathbf{E}[e^{s(N+X_1+\cdots+X_N)}|N]] = \mathbf{E}[\mathbf{E}[e^{sN} \cdot e^{s(X_1+\cdots+X_N)}|N]] = \mathbf{E}[e^{sN}(M_X(s))^N]$
$= \mathbf{E}[(e^s M_X(s))^N] = M_N(s)|_{e^s = e^s M_X(s)}.$

$M_N(s) = \frac{(2/3)e^s}{1-(1/3)e^s}.$

$M_X(s) = e^{2s^2+2s}.$

$M_Z(s) = \frac{(2/3)e^s e^{2s^2+2s}}{1-(1/3)e^s e^{2s^2+2s}} = \frac{2e^{2s^2+3s}}{3-e^{2s^2+3s}}.$