MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2004)

Final Exam Solutions

Problem 2

2(a) The spam arrival rate is 8 messages per hour, so the number of spam emails in a ten hour period is a Poisson random variable with parameter $\lambda = 80$.

$$\mathbf{P}\left(\text{no spam}\right) = e^{-80}\frac{80^0}{0!} = e^{-80} \approx 1.8 \cdot 10^{-35}.$$

2(b) Let Y_3 be the arrival time of the third spam email. Y_3 is an Erlang random variable of order $k = 3$ with parameter $\lambda_s = 8$.

$$\mathbf{E}\left[Y_3\right] = \frac{k}{\lambda_s} = \frac{3}{8}.$$

$$\text{var}(Y_3) = \frac{k}{\lambda_s^2} = \frac{3}{64}.$$

2(c) Let R and S denote the number of regular and spam messages received, respectively. Let T_1, T_2, … be the processing times for the regular messages and T be the total processing time, both in seconds. Then

$$T = 2S + T_1 + T_2 + \cdots + T_R,$$

and the question asks for $\mathbf{E}[T]$ and $\mathrm{var}(T)$.

Using elementary properties of sums of independent random variables and the standard formulas for sums of random numbers of independent random variables gives

$$\begin{aligned}\mathbf{E}[T] &= 2\mathbf{E}[S] + \mathbf{E}[T]\,\mathbf{E}[R] \\ &= 2 \cdot 80 + 90 \cdot 20 \;=\; 1960\end{aligned}$$

and

$$\begin{aligned}\mathrm{var}(T) &= 4\mathrm{var}(S) + \mathrm{var}(T)\mathbf{E}[R] + (\mathbf{E}[T])^2\mathrm{var}(R) \\ &= 4 \cdot 80 + \frac{(120-60)^2}{12} \cdot 20 + 90^2 \cdot 20 \;=\; 168320\end{aligned}$$

2(d) When we consider regular and spam email together, we have a Poisson arrival process at rate 10 messages per hour where each arrival has probability $\frac{2}{2+8} = \frac{1}{5}$ of being a regular message.

$$\mathbf{P}(\text{party invitation}) = \mathbf{P}(\text{regular mail}) \cdot \mathbf{P}(\text{party invitation} \mid \text{regular mail}) = \frac{1}{5} \cdot \frac{1}{20} = \frac{1}{100}.$$

2(e) Party invitations arrive as a Poisson process with rate $\lambda_p = 0.1$ messages per hour. The number of party invitations to arrive in a 10 hour period is thus a Poisson random variable with parameter 1. The desired PMF is

$$p(k) = \begin{cases} e^{-1}\frac{1^k}{k!}, & k = 0, 1, 2, \ldots; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{e}\frac{1}{k!}, & k = 0, 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

2(f) For this and the next part, we use the merged Poisson process that has both regular and spam emails. Time is irrelevant to us; what matters is that each arrival is a regular message with probability $\frac{1}{5}$ (and spam with probability $\frac{4}{5}$).

This part asks for the expected number of spam messages before the first regular mail arrival. Let X be the number of messages up to and including the first regular mail message. X is a geometric random variable with parameter $\frac{1}{5}$. Thus $\mathbf{E}[X] = 5$. The answer is thus 4.

2(g) Let S be the number of spam messages received out of the first 100 total messages. S is a binomial($100,\frac{4}{5}$) random variable so

$$\mathbf{P}(S=80) = \binom{100}{80}\left(\frac{4}{5}\right)^{80}\left(\frac{1}{5}\right)^{20}.$$

As a precursor to the CLT-based approximation, we compute $\mathbf{E}[S] = 100\cdot\frac{4}{5} = 80$ and $\text{var}(S) = 100\cdot\frac{4}{5}\cdot\frac{1}{5} = 16$. Now we can compute

$$\begin{aligned}
\mathbf{P}(S=80) &= \mathbf{P}(79.5 \le S \le 80.5) \\
&= \mathbf{P}\left(\frac{79.5-80}{4} \le \frac{S-80}{4} \le \frac{80.5-80}{4}\right) \\
&= \mathbf{P}\left(-0.125 \le \frac{S-80}{4} \le 0.125\right) \\
&\approx \Phi(0.125) - \Phi(-0.125) \quad \text{since } (S-80)/4 \text{ is approximately standard normal} \\
&= 2\Phi(0.125) - 1 \\
&\approx 2\cdot 0.5497 - 1 \ = \ 0.0995
\end{aligned}$$

2(h) Let T be the time asleep (in hours) and M be the number of messages that arrive while asleep. We are given that T is an exponential(1) random variable. Given $T = t$, M is a Poisson random variable with parameter $10t$. Now a version of Bayes's Rule gives (for $t \geq 0$)

$$\begin{aligned} f_{T\mid\{M=0\}}(t) &= \frac{p_{M|T}(0 \mid t) f_T(t)}{p_M(0)} = \frac{p_{M|T}(0 \mid t) f_T(t)}{\int_0^\infty p_{M|T}(0 \mid t) f_T(t)\,dt} \\ &= \frac{e^{-10t}e^{-t}}{\int_0^\infty e^{-10t}e^{-t}\,dt} = \frac{e^{-11t}}{\int_0^\infty e^{-11t}\,dt} \\ &= \frac{e^{-11t}}{-\frac{1}{11}[e^{-11t}]_0^\infty} = 11e^{-11t}. \end{aligned}$$

To give a complete answer, we should also note that

$$f_{T\mid\{M=0\}}(t) = 0 \qquad \text{for } t < 0.$$

Problem 3

3(a) All three states are recurrent. There is a single recurrent class which is aperiodic.

Comments: Each of three parts should be all or nothing. Don't require reasoning.
+2: Recurrent states are 1, 2, 3
+2: Transient states are none
+1: Aperiodic: YES

3(b) Use the Law of Total Probability and $\mathbf{P}(X_n = i) \approx \pi_i$:

$$\begin{aligned}\mathbf{P}(X_n = X_{n+1}) &= \mathbf{P}(X_n = X_{n+1} \mid X_n = 1)\,\mathbf{P}(X_n = 1) \\ &\quad + \mathbf{P}(X_n = X_{n+1} \mid X_n = 2)\,\mathbf{P}(X_n = 2) \\ &\quad + \mathbf{P}(X_n = X_{n+1} \mid X_n = 3)\,\mathbf{P}(X_n = 3) \\ &\approx \frac{1}{2}\cdot\pi_1 + \frac{1}{3}\cdot\pi_2 + \frac{1}{2}\cdot\pi_3.\end{aligned}$$

3(c) We need three linearly independent equations involving unknowns π_1, π_2, and π_3. The standard technique for this is to have the normalization equation and any two balance equations:

$$\begin{aligned}\pi_1 + \pi_2 + \pi_3 &= 1 \\ -\frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 &= 0 \\ \frac{1}{3}\pi_2 - \frac{1}{2}\pi_3 &= 0\end{aligned}$$

Solving gives $\pi_1 = \frac{4}{9}$, $\pi_2 = \frac{1}{3}$, $\pi_3 = \frac{2}{9}$.

3(d) Define t_i as the expected time to first enter state 3 starting from state i, $i = 1, 2, 3$.

$$\begin{aligned} t_1 &= 1 + \frac{1}{2}t_1 + \frac{1}{2}t_2 \\ t_2 &= 1 + \frac{1}{3}t_1 + \frac{1}{3}t_2 + \frac{1}{3}t_3 \\ t_3 &= 0 \end{aligned}$$

Solving gives $t_1 = 7$, $t_2 = 5$, $t_3 = 0$. The final answer is now obtained with the Law of Total Expectation by conditioning on the possible initial states:

$$\begin{aligned} \mathbf{E}\left[\text{time to first enter state 3}\right] &= \mathbf{E}\left[\text{time to first enter state 3} \mid X_0 = 1\right]\mathbf{P}\left(X_0 = 1\right) \\ &\quad + \mathbf{E}\left[\text{time to first enter state 3} \mid X_0 = 2\right]\mathbf{P}\left(X_0 = 2\right) \\ &= t_1 \cdot \frac{1}{3} + t_2 \cdot \frac{2}{3} = \frac{17}{3}. \end{aligned}$$

3(e)(i) Because of the Markov property, T_1, T_2, ... are identically distributed. Thus there can only be convergence in probability if there is some constant a such that $\mathbf{P}(T_1 = a) = 1$. This is obviously not the case because there are so many distinct state sequences that start in 3 and end when they return to 3 for the first time (3123, 312123, 31212123, ...).

3(e)(ii) Since the T_is are identically distributed and $\mathbf{E}[T_i]$ is finite, the convergence in probability of Q_n follows from the weak law of large numbers. This convergence in probability is to the value $\mathbf{E}[T_i]$.

The question does not require the calculation of $\mathbf{E}[T_i]$, but we can find $\mathbf{E}[T_i]$ using the t_is computed above and the Law of Total Expectation (where the conditioning is on the three possible transitions from state 3):

$$\begin{aligned}
\mathbf{E}[T_i] &= \mathbf{E}[\text{time to return to 3} \mid \text{next state is 1}]\,\mathbf{P}(\text{next state is 1}) \\
&\quad + \mathbf{E}[\text{time to return to 3} \mid \text{next state is 2}]\,\mathbf{P}(\text{next state is 2}) \\
&\quad + \mathbf{E}[\text{time to return to 3} \mid \text{next state is 3}]\,\mathbf{P}(\text{next state is 3}) \\
&= (1+t_1)\cdot\frac{1}{2} + t_2\cdot 0 + 1\cdot\frac{1}{2} \\
&= \frac{9}{2}.
\end{aligned}$$

3(f)(i) This part and the final part are very loosely connected to the above parts. The only fact that is carried forward is the PMF of X_0.

Using a continuous version of Bayes's Rule:

$$\begin{aligned}
\mathbf{P}(X_0 = 1 \mid Y = y) &= \frac{\mathbf{P}(X_0=1)\, f_{Y|\{X_0=1\}}(y)}{\mathbf{P}(X_0=1)\, f_{Y|\{X_0=1\}}(y) + \mathbf{P}(X_0=2)\, f_{Y|\{X_0=2\}}(y)} \\
&= \frac{\frac{1}{3}f(y-1)}{\frac{1}{3}f(y-1)+\frac{2}{3}f(y-2)} \quad \text{where } f(\cdot) \text{ is the standard normal PDF} \\
&= \frac{\frac{1}{3}e^{-(y-1)^2/2}}{\frac{1}{3}e^{-(y-1)^2/2}+\frac{2}{3}e^{-(y-2)^2/2}} \\
&= \frac{1}{1+2e^{(2y-3)/2}}.
\end{aligned}$$

3(f)(ii) Conditioned on $X_0 = 1$, Y is a normal random variable with mean 1 and variance 1.

Let $Z = W^2$ where W is a normal random variable with mean 1 and variance 1. The standard approach to finding the PDF of Z is to first find the CDF and then differentiate:

$$\begin{aligned}
F_Z(z) &= \mathbf{P}\left(Z \le z\right) \\
&= \mathbf{P}\left(W^2 \le z\right) \\
&= \mathbf{P}\left(-\sqrt{z} \le W \le \sqrt{z}\right) \quad \text{for } z > 0 \text{ (otherwise 0)} \\
&= F_W(\sqrt{z}) - F_W(-\sqrt{z}) \quad \text{for } z > 0 \text{ (otherwise 0)}
\end{aligned}$$

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \frac{1}{2\sqrt{z}} f_W(\sqrt{z}) + \frac{1}{2\sqrt{z}} f_W(-\sqrt{z}) \quad \text{for } z > 0 \text{ (otherwise 0)} \\
&= \frac{1}{2\sqrt{z}} \left(\frac{1}{\sqrt{2\pi}} e^{-(\sqrt{z}-1)^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{z}-1)^2/2} \right) \quad \text{for } z > 0 \text{ (otherwise 0)} \\
&= \frac{1}{2\sqrt{2\pi z}} \left(e^{-(\sqrt{z}-1)^2/2} + e^{-(\sqrt{z}+1)^2/2} \right) \quad \text{for } z > 0 \text{ (otherwise 0)}
\end{aligned}$$

Finally,

$$f_{Y^2|\{X_0=1\}}(z) = f_Z(z) = \begin{cases} \frac{1}{2\sqrt{2\pi z}} \left(e^{-(\sqrt{z}-1)^2/2} + e^{-(\sqrt{z}+1)^2/2} \right), & \text{for } z > 0; \\ 0, & \text{otherwise.} \end{cases}$$