Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2005)

Problem Set 8 - Solutions
Due: April 20, 2005

1. (a) At any point of transmission, we only care if we get a codeword with weight 0 or a codeword with weight at least 2. For any other codewords, the transmission will continue and we return to the original problem. Therefore, the problem becomes: given a codeword with weight 0 or a codeword with weight at least 2, what is the probability that it is a codeword with weight at least 2.

$$\begin{aligned}
&\mathbf{P}(\text{weight 2 or weight 3}|\text{weight 0 or (weight 2 or weight 3)}) \\
&= \frac{\mathbf{P}(\text{weight 2 or weight 3})}{\mathbf{P}(\text{weight 0}) + \mathbf{P}(\text{weight 2 or weight 3})} \\
&= \frac{4/8}{4/8 + 1/8} \\
&= \frac{4}{5}
\end{aligned}$$

(b) In this case, since the codewords are independent, the given information is irrelevant. $\mathbf{P}(\text{next 2 codewords have weight 0}) = (\frac{1}{8})^2 = \frac{1}{64}$.

(c) i. The third order Pascal PMF for random variable N, as defined in the text, is the probability the third success comes on the n^{th} trial. Thus, the random variable, K, is a shifted version of the third order Pascal PMF, i.e. $K = N - 1$. So, the probability that 2 timing pulses are in the first k codewords, where the next codeword will be the third timing pulse, can be expressed as:

$$p_K(k) = \binom{k}{2} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{k-2}, \quad k \geq 2.$$

ii. L is a binomial random variable, with $n = 100$, $p = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, so $E[L] = np = 25$.

iii. The number of "0"s before the first timing pulse, M, can be written as a random sum:

$$M = X_1 + X_2 + \cdots + X_N,$$

where X_i is the number of "0"s that occur on codeword i, and N is the number of codewords before the first timing pulse). We notice that X is equally likely to be either 1 or 2, and that N is a shifted geometric: $N = R - 1$, where R is a geometric random variable with parameter $\frac{1}{4}$. Now we can apply our random sum formulae.

$$\mathbf{E}[M] = \mathbf{E}[X]\mathbf{E}[N] = (\frac{3}{2})(4-1) = \frac{9}{2}$$

$$\text{var}(M) = \mathbf{E}[N]\text{var}(X) + (\mathbf{E}[X])^2\text{var}(N) = (4-1)(\frac{1}{4}) + (\frac{3}{2})^2(12) = \frac{111}{4}.$$

iv. The probability of a codeword with 3 "1"s is $\frac{1}{8}$, so the number of them in 100 codewords is a binomial random variable with parameters $\frac{1}{8}$ and 100.

$$p_N(n) = \binom{100}{n} \left(\frac{1}{8}\right)^n \left(\frac{7}{8}\right)^{100-n}, \quad n = 0, \ldots, 100.$$

v. Given we have l timing pulse, the probability of each timing pulse being "111" is equal to the probability of it being "000". Therefore, we have a binomial random variable with parameters $\frac{1}{2}$ and l.

$$p_{N|L}(n|l) = \binom{l}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{l-n} = \binom{l}{n}\left(\frac{1}{2}\right)^l, \quad n = 0, \ldots, l, \quad l = 0, \ldots, 100.$$

(d) i. Q, the number of codewords, can be expressed as the sum of 3 independent random variables, X, Y, and Z. X is the number of codewords until the first switch, Y the number of additional codewords until the second switch, and Z the additional number until the third switch. We see that X is a geometric random variable with parameter $\frac{1}{8}$, Y is geometric with parameter $\frac{1}{4}$, and Z geometric with parameter $\frac{1}{2}$. Hence,

$$\mathbf{E}[Q] = \mathbf{E}[X] + \mathbf{E}[Y] + \mathbf{E}[Z] = 8 + 4 + 2 = 14.$$

ii. The transform of the sum of independent random variables is the product of their transforms, so

$$M_Q(s) = M_X(s)M_Y(s)M_Z(s) = \left(\frac{\frac{1}{8}e^s}{1-\frac{7}{8}e^s}\right)\left(\frac{\frac{1}{4}e^s}{1-\frac{3}{4}e^s}\right)\left(\frac{\frac{1}{2}e^s}{1-\frac{1}{2}e^s}\right).$$

2. (a) The random veriable R is binomial with parameters p and n. Hence,

$$p_R(r) = \binom{n}{r}(1-p)^{n-r}p^r, \qquad \text{for } r = 0, 1, 2, ..., n,$$

$\mathbf{E}[R] = np$, and $\text{var}(R) = np(1-p)$.

(b) Let A be the event that the first photon ends up being the only one sent to its port. This event is the union of two disjoint events:

i. the first photon is sent to port A and the remaining photons are sent to port B, and,

ii. the first photon is sent to port B and the remaining photons are sent to port A.

$$\mathbf{P}(A) = p(1-p)^{n-1} + (1-p)p^{n-1}$$

(c) Let B be the event that at least one port ends up with a total of exactly one photon. The event B occurs if exactly one or both of the ports end up with exactly 1 photon, so

$$\mathbf{P}(B) = \begin{cases} 1, & \text{if } n = 1, \\ 2p(1-p), & \text{if } n = 2, \\ \binom{n}{1}(1-p)^{n-1}p + \binom{n}{n-1}p^{n-1}(1-p), & \text{if } n = 3, 4, 5, ... \end{cases}$$

(d) $D = R - G = R - (n - R) = 2R - n$. We have $\mathbf{E}[D] = 2\mathbf{E}[R] - n = 2np - n$. Since $D = 2R - n$, and n is a constant,

$$\text{var}(D) = 4\text{var}(R) = 4np(1-p).$$

(e) Let C be the event that each of the first 2 photons is sent to port A. Let X_i be a Bernoulli random variable which is 1 if the ith photon is sent to port A, and 0 if not. Given that C occurred, the random variable R becomes

$$2 + X_3 + X_4 + \cdots + X_n.$$

Hence,

$$\begin{aligned}\mathbf{E}[R|C] &= \mathbf{E}[2 + X_3 + X_4 + \cdots + X_n] \\ &= 2 + (n-2)\mathbf{E}[X_i] \\ &= 2 + (n-2)p.\end{aligned}$$

Similarly, the conditional variance of R is

$$\begin{aligned}\text{var}[R|C] &= \text{var}[2 + X_3 + X_4 + \cdots + X_n] \\ &= (n-2)\text{var}[X_i] \\ &= (n-2)p(1-p).\end{aligned}$$

Finally, given that the first two photons are sent to port A, the probability that a total of r photons are sent to port A is equal to the probability that $r-2$ of the remaining $n-2$ photons are sent to port A:

$$p_{R|C}(r) = \binom{n-2}{r-2}(1-p)^{n-r}p^{r-2}, \quad \text{for } r = 2, ..., n.$$

3. (a) i. Since it is implied that each second is independent from each other (i.e. if a mosquito landed on your neck last second, it doesn't affect the likelihood of one landing this or next second), the PMF for the time until the first mosquito lands is simply a geometric random variable. In this random variable, the "success" event is when a mosquito lands with probability 0.2, with the "failure" event being 0.8.

$$p_T(t) = \begin{cases} (.8)^{t-1}(.2) & t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

ii. The expected time of the geometric PMF with parameter p is $\frac{1}{p}$, so the expected time until the first mosquito lands on you is $\frac{1}{.2} = 5$ seconds.

iii. The scenario in which mosquitoes independently land on you each second can be modeled as a Bernoulli process, with each Bernoulli trial being the event that a mosquito lands on you on a given second. Because the Bernoulli process is memoryless, it doesn't matter whether or not you were bitten for the first one, ten, or two hundred seconds. Thus, the expected time from $T = 10$ is identical to the answer in the previous part, $\frac{1}{.2} = 5$ seconds.

(b) i. Because the PDF that models the time until the first mosquito arrives is exponential, the expected time until it lands is $\frac{1}{.2} = 5$ seconds.

ii. It has been previously shown that the exponential PDF exhibits the memorylessness property. In other words, looking at an exponential PDF from some future time $\neq 0$ will still yield an exponential PDF with the same parameter. Thus, the expected time from $T = 10$ is identical to the answer in the previous part, $\frac{1}{.2} = 5$ seconds.

4. Let X_i be a Bernoulli random variable which is 1 if the ith photon is sent to port A, and 0 if not. Then, the number of photons sent to port A is

$$R = X_1 + \cdots + X_N.$$

We are dealing here with the sum of a random number of independent random variables. As discussed in Section 4.4, the transform associated with R is found by starting with the transform associated with N, which is

$$M_N(s) = e^{\lambda(e^s - 1)},$$

and replacing each occurrence of e^s by the transform associated with X_i, which is

$$M_X(s) = 1 - p + pe^s.$$

We obtain

$$M_R(s) = e^{\lambda(1-p+pe^s-1)} = e^{p\lambda(e^s-1)}.$$

We observe that this is the transform of a Poisson random variable with parameter λp, thus verifying our earlier statement for the PMF of R.

5. The left hand side corresponds to the probability that the time of 6th arrival is greater than T, which is same as the probability that the number of arrivals up to time T is less than or equal to 5. Therefore $a = 0$ and $b = 5$ on the right hand side.