Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Problem Set 9
Due: April 27, 2005

1. At time $t = 0$, you start getting e-mail messages according to a Poisson process of rate λ. Each e-mail is independently spam with probability p, an invitation to a party with probability q, or neither with probability $1 - p - q$. Each spam message is immediately deleted, each party invitation is immediately directed to a folder labeled IMPORTANT, and other messages immediately go to a folder named OTHER. We can assume that the process of getting emails goes on indefinitely. Starting with empty folders at $t = 0$, if 10 messages get collected in the IMPORTANT folder before 10 messages get collected in the OTHER folder, you will conclude that you are "very popular". Otherwise you conclude that you are not very popular.

   (a) Find the CDF of T, the time at which you decide whether or not you are "very popular". (Your answer can contain summation(s) of terms.)

   (b) How many of the first 19 non-spam messages must be party invitations for you to conclude that you are very popular?

   (c) What is the probability that you end up deciding you are very popular? (Hint: consider the ongoing process of e-mails and use your answer to the previous part.)

2. Rasheed Wallace is a very good, but emotionally unbalanced, basketball player. We model this as follows (allowing scores to be non-integral):

   - For each minute that he plays, his team outscores its opponent by 0.25 points. For each minute that he does not play, his team is outscored by 0.5 points.
   - He is charged with personal fouls in a time-homogenous manner at a rate of one foul per eight minutes. On other words, his personal fouls form a continuous-time arrival process that is a Poisson process with parameter $\lambda = \frac{1}{8}$ (with all times measured in minutes).
   - Any time he commits a sixth personal foul in a game, he loses his temper, gets three technical fouls, and is ejected from the game; this costs his team 6 points in addition to the fact that he does not play the remainder of the game.

   Except where indicated below, assume Wallace plays from the beginning of the game until the end of the game or ejection. Basketball games are 48 minutes long and since we have a continuous (non-integral) model for scores we need not consider the possibility of a tie at the end of 48 minutes.

   (a) Let X_t denote the number of personal fouls Wallace has committed t minutes from the beginning of the game. Determine the PMF of X_t.

   (b) What is the probability that Wallace fouls out of a game?

   (c) What is the probability that Wallace's team wins?

   (d) To help Wallace rehabilitate his image, his new coach Larry Brown decides to remove Wallace—immediately and for the rest of the game—whenever he commits a fifth foul in a game. (Wallace thus nevers fouls out, nor loses his temper, nor gets technical fouls.) Now what is the probability that Wallace's team wins?

3. A discrete-time Markov chain with seven states has the following transition probabilities:

$$p_{ij} = \begin{cases} 0.5 & , \quad (i,j) = (3,2), (3,4), (5,6) \text{ and } (5,7) \\ 1 & , \quad (i,j) = (1,3), (2,1), (4,5), (6,7) \text{ and } (7,5) \\ 0 & , \quad \text{otherwise} \end{cases}$$

   In the questions below, we let X_k be the state of the Markov process at time k.

   (a) Give a pictorial representation of the discrete-time Markov chain.

   (b) For what values of n is the probability $r_{15}(n) = \mathbf{P}(X_n = 5 \mid X_0 = 1) > 0$?

   (c) What is the set of states $A(i)$ that is accessible from state i, for each $i = 1, 2, \ldots, 7$?

   (d) Identify which states are transient and which states are recurrent. For each recurrent class, state whether it is periodic (and give the period) or aperiodic.

   (e) What is the minimum number of transitions with nonzero probability that must be added so that all seven states form a single recurrent class?

4. Out of the d doors of my house, suppose that in the beginning $k > 0$ are unlocked and $d - k$ are locked. Every day, I use exactly one door, and I am equally likely to pick any of the d doors. At the end of the day, I leave the door I used that day locked.

   (a) Show that the number of unlocked doors at the end of day n, L_n, evolves as the state in a Markov process for $n \geq 1$. Write down the transition probabilities p_{ij}.

   (b) List transient and recurrent states.

   (c) Is there an absorbing state? How does $r_{ij}(n)$ behave as $n \to \infty$?

   (d) Now, suppose that each day, if the door I pick in the morning is locked, I will leave it unlocked at the end of the day, and if it is initially unlocked, I will leave it locked. Repeat parts (a)-(c) for this strategy.

   (e) My third strategy is to alternate between leaving the door I use locked one day and unlocked the next day (regardless of the initial condition of the door.) In this case, does the number of unlocked doors evolve as a Markov chain, why/why not?