Solutions of Recitation 16
April 12, 2005
Review of Bernoulli and Poisson Processes

1. (a) i. True. The event $N_1 = N_2$ means $X_2 = 0$ and the event $N_3 = N_4$ means $X_4 = 0$. The random variable X_2 and X_4 are i.i.d.

   ii. False. The random variable Y is a Binomial random variable with parameter p and $n = 5$.

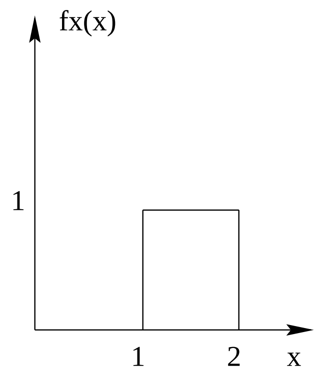
   (b) i. False. $p_{N_t}(k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$, $k = 0, 1, ...$ and $p_X(k) = e^{-1.5\lambda}\frac{(1.5\lambda)^k}{k!}$, $k = 0, 1, ...$ Hence, $\text{var}(X) = 1.5\lambda = 3$.

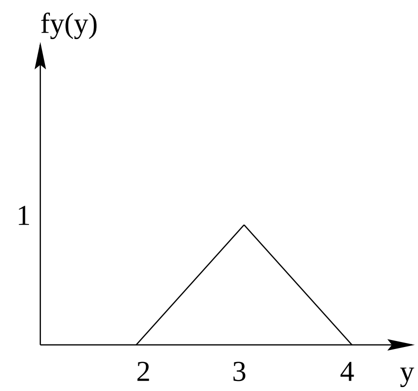
   ii. False. Because $N_4 = (N_4 - N_{3.2}) + X + N_{1.7}$, $\text{cov}(N_4, X) = \mathbf{E}[(N_4 - \mathbf{E}[N_4])(X - \mathbf{E}[X])] = \text{var}(X) = 3$. They are not independent.

2. The solution is on page 303 of the textbook.

3. Let X_i be the ith interarrival interval and let Y_i be the arrival time of the ith bus.

   (a) Because the interarrival times X_1 and X_2 are independent, we can convolve to find the PDF of $Y_2 = X_1 + X_2$.





   (b) $Y_4 = X_1 + X_2 + X_3 + X_4$.
   Thus $\mathbf{E}[Y_4] = \mathbf{E}[X_1 + X_2 + X_3 + X_4] = 4\mathbf{E}[X] = 4(1.5) = 6$ hours.
   Therefore, the expected arrival time of the fourth bus is 6pm.

   (c) The following events, which form a partition of the sample space, describe the different ways that the first 4 buses after 12pm may arrive:
   A_1: 4 one-bus arrivals
   A_2: 3 one-bus arrivals followed by 1 two-bus arrival
   A_3: 2 one-bus arrivals and 1 two-bus arrivals
   A_4: 1 one-bus arrival and 2 two-bus arrivals
   (the one-bus arrival occurs before the second two-bus arrival)
   A_5: 2 two-bus arrivals

   Using the total expectation theorem, we have:

$$\begin{aligned}\mathbf{E}[Y_4] &= \mathbf{E}[Y_4 \mid A_1]\mathbf{P}(A_1) + \mathbf{E}[Y_4 \mid A_2]\mathbf{P}(A_2) + \mathbf{E}[Y_4 \mid A_3]\mathbf{P}(A_3) \\ &\quad + \mathbf{E}[Y_4 \mid A_4]\mathbf{P}(A_4) + \mathbf{E}[Y_4 \mid A_5]\mathbf{P}(A_5)\end{aligned}$$

To find the terms in the right-hand side of the above equation, we first find $\mathbf{P}(B_1)$, $\mathbf{P}(B_2)$, $\mathbf{E}[X \mid B_1]$, and $\mathbf{E}[X \mid B_2]$, where B_i is the event that an arrival has i buses.

$$\begin{aligned}
\mathbf{P}(B_2) &= \mathbf{P}(B_2 \mid X > 1.5)\mathbf{P}(X > 1.5) + \mathbf{P}(B_2 \mid X \leq 1.5)\mathbf{P}(X \leq 1.5) \\
&= \frac{1}{2} \cdot \frac{1}{2} + 0 = \frac{1}{4} \\
\mathbf{P}(B_1) &= 1 - \mathbf{P}(B_2) = \frac{3}{4}
\end{aligned}$$

To find $\mathbf{E}[X \mid B_1]$ and $\mathbf{E}[X \mid B_2]$, we first find the conditional PDFs for X using a continuous version of Bayes' rule, then calculate the conditional expectation using the corresponding conditional PDF:

$$\begin{aligned}
f_{X|B_1}(x) &= \frac{\mathbf{P}(B_1 \mid X = x) f_X(x)}{\mathbf{P}(B_1)} \\
&= \begin{cases} \frac{1 \cdot 1}{\frac{3}{4}} = \frac{4}{3}, & \text{for } 1 \leq x \leq \frac{3}{2}; \\ \frac{\frac{1}{2} \cdot 1}{\frac{3}{4}} = \frac{2}{3}, & \text{for } \frac{3}{2} \leq x \leq 2; \\ 0, & \text{otherwise} \end{cases} \\
\mathbf{E}[X \mid B_1] &= \frac{2}{3} \cdot \frac{5}{4} + \frac{1}{3} \cdot \frac{7}{4} = \frac{17}{12} \\
f_{X|B_2}(x) &= \frac{\mathbf{P}(B_2 \mid X = x) f_X(x)}{\mathbf{P}(B_2)} \\
&= \begin{cases} \frac{\frac{1}{2} * 1}{\frac{1}{4}} = 2, & \text{for } \frac{3}{2} \leq x \leq 2; \\ 0, & \text{otherwise} \end{cases} \\
\mathbf{E}[X \mid B_2] &= \frac{7}{4}
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
\mathbf{E}[Y_4] &= \mathbf{E}[Y_4 \mid A_1]\mathbf{P}(A_1) + \mathbf{E}[Y_4 \mid A_2]\mathbf{P}(A_2) + \\
&\quad \mathbf{E}[Y_4 \mid A_3]\mathbf{P}(A_3) + \mathbf{E}[Y_4 \mid A_4]\mathbf{P}(A_4) + \mathbf{E}[Y_4 \mid A_5]\mathbf{P}(A_5) \\
&= \left(4 \cdot \frac{17}{12}\right)\left(\frac{3}{4}\right)^4 + \left(3 \cdot \frac{17}{12} + \frac{7}{4}\right)\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) + \left(2 \cdot \frac{17}{12} + \frac{7}{4}\right) 3 \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) \\
&\quad + \left(\frac{17}{12} + 2 \cdot \frac{7}{4}\right) 2 \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \left(2 \cdot \frac{7}{4}\right)\left(\frac{1}{4}\right)^2 \\
&= \frac{645}{128} \text{ hours}
\end{aligned}$$

or $\mathbf{E}[Y_4] = 5{:}02{:}20$ (5 pm, 2 minutes, 20 seconds)

(d) Let Z be the number of arrivals up to and including the first two-bus arrival. Since we found that $\mathbf{P}(B_2) = \frac{1}{4}$, Z is geometrically distributed with probability of success $\frac{1}{4}$.

Let $\tilde{X}_i$ be the interarrival time between two-bus arrivals and let $\tilde{Y}_i$ be the arrival time of the ith two-bus arrival. We know $\tilde{X}_1$ will be the sum of the interarrival times of the first $Z-1$ arrivals, which are all one-bus arrivals, and the interarrival time of the Zth arrival, which is the first two-bus arrival.

$$\begin{aligned}
\mathbf{E}[\tilde{Y}_4] &= \mathbf{E}[\tilde{X}_1 + \tilde{X}_2 + \tilde{X}_3 + \tilde{X}_4] \\
&= 4\mathbf{E}[\tilde{X}] \\
&= 4(\mathbf{E}[X \mid B_1]\mathbf{E}[Z-1] + \mathbf{E}[X \mid B_2]) \\
&= 4\left(\frac{17}{12}(4-1) + \frac{7}{4}\right) = 24 \text{ hours}
\end{aligned}$$

Therefore, the expected arrival time of the fourth two-bus arrival is 12pm the next day.