6.041 Fall 2004 Final
Tuesday 14th of December, 1:30-4:30pm
DO NOT TURN THIS QUIZ OVER UNTIL YOU ARE TOLD TO DO SO

Your Full Name:	
Recitation Instructor and TA's Names:	

- This quiz has 3 problems, worth 100 points. Parts are not necessarily in order of difficulty.
- Write your solutions in the space provided in this quiz itself. We will not consider any work elsewhere. Blue books are provided for your scratch work. Hand in both this quiz and the blue books.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for three double-sided, 8.5 by 11 formula sheets.

Problem		**Your Score**	**Problem**		**Your Score**
1 (1 point)			**3 (50 points)**		
2 (49 points)			**3.(a)**	**5**	
2.(a)	**5**		**3.(b)**	**7**	
2.(b)	**5**		**3.(c)**	**7**	
2.(c)	**7**		**3.(d)**	**7**	
2.(d)	**5**		**3.(e)(i)**	**5**	
2.(e)	**6**		**3.(e)(ii)**	**5**	
2.(f)	**7**		**3.(f)(i)**	**7**	
2.(g)	**7**		**3.(f)(ii)**	**7**	
2.(h)	**7**		**Total (100 points)**		

Problem 1. (1 point)

Write the names of your recitation instructor **and** TA on the front of this quiz and the blue book.

Problem 2. (49 points)

Regular emails arrive according to a Poisson process with rate $\lambda_r = 2$ messages per hour. Spam email arrives according to a Poisson process with rate $\lambda_s = 8$ messages per hour, independently of the regular emails. Each regular, non-spam, email is an invitation to a party with probability $p = 0.05$, independently of everything else.

(5 pts) **2(a)** Find the probability that you will get no spam emails from noon to 10pm on a particular day.

$\mathbf{P}$(no spam from noon to 10pm)=______________________________________

Show your reasoning and calculations here:

(5 pts) **2(b)** Find the expectation and the variance of the arrival time of the third spam message.

E(arrival time of 3rd spam)=____

var(arrival time of 3rd spam)=____

Show your reasoning and calculations here:

(7 pts) **2(c)** It takes you 2 seconds to recognize and delete a spam message. The time it takes you to read and answer regular emails is uniformly distributed between 60 and 120 seconds. Find the expectation and the variance of the time it will take you to deal with all the email you receive from noon to 10pm.

E(time to deal with email from noon to 10pm)=________________________________

var(time to deal with email from noon to 10pm)=________________________________

Show your reasoning and calculations here:

(5 pts) **2(d)** You just got a new email. What is the probability that it is a party invitation?

P(new email is a party invitation)=______________________________________

Show your reasoning and calculations here:

(6 pts) **2(e)** Find the PMF of the number of party invitations you will receive from noon to 10pm.

____

Show your reasoning and calculations here:

(7 pts) **2(f)** You just got back from classes and checked your email. Find the expectation of the number of spam messages that will arrive between now and the next regular email.

E(number of spam between now and next regular email)=______________________________

Show your reasoning and calculations here:

(7 pts) **2(g)** We are interested in the probability that out of the first 100 emails you receive, exactly 80 are spam. Write down the exact expression for the probability and find a *good* approximation of it (we are looking for a number; writing down a formula is insufficient).

P(exactly 80 of the first 100 emails are spam)=__________________________(exact expr.)

P(exactly 80 of the first 100 emails are spam)≈__________________________(a good approx.)

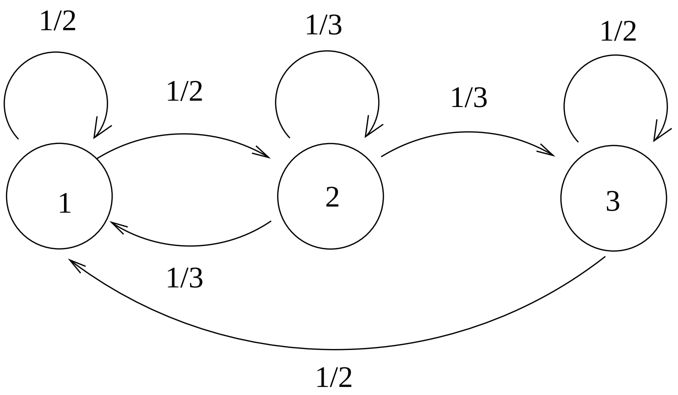
Show your reasoning and calculations here:

(7 pts) **2(h)** Right after you checked your email, you fell asleep for a random period of time. The time asleep in hours is described by an exponential random variable with parameter 1, and is independent of the email arrival processes. You wake up and find that no emails have arrived. Find the conditional PDF of your sleeping time given this information.

____

Show your reasoning and calculations here:

Problem 3. (50 points)



Let X_n be a Markov chain with transition probabilities as shown in the figure. The chain starts in state 1 with probability 1/3, and in state 2 with probability 2/3.

(5 pts) **3(a)** Identify recurrent states and transient states. Identify recurrent classes. Is the Markov chain aperiodic?

Recurrent states:____

Transient states:____

Aperiodic (circle one): YES NO

(7 pts) **3(b)** Let π_i be the steady-state probability of state i. If n is a large number, find a good approximation for the probability that $X_n = X_{n+1}$. Your answer may involve the — yet unknown — variables π_i.

$\mathbf{P}(X_n = X_{n+1}) \approx$ ____ (a good approximation)

Show your reasoning and calculations here:

(7 pts) **3(c)** Find the steady-state probabilities π_i.

π_1=______________

π_2=______________

π_3=______________

Show your reasoning and calculations here:

(7 pts) **3(d)** Find the expected time until the chain enters state 3 for the first time.

E(time until the chain enters state 3 for the first time)=____

Show your reasoning and calculations here:

3(e) Let Y_n be the time at which state 3 is visited for the n-th time, let $T_n = Y_{n+1} - Y_n$ be the time between consecutive visits to state 3, and let $\mu = E[T_n]$, assumed to be available.

(5 pts) **3(e)(i)** Does the sequence T_n converge in probability? If yes, to what value? If no, briefly explain why.

Converges (circle one): YES, to ____________ NO

Show your reasoning and calculations here:

(repeated) **3(e)** Let Y_n be the time at which state 3 is visited for the n-th time, let $T_n = Y_{n+1} - Y_n$ be the time between consecutive visits to state 3, and let $\mu = E[T_n]$, assumed to be available.

(5 pts) **3(e)(ii)** Does the sequence $Q_n = \frac{1}{n}\sum_{i=1}^{n} T_i$ converge in probability? If yes, to what value? If no, briefly explain why.

Converges (circle one): YES, to ____________ NO

Show your reasoning and calculations here:

3(f) At the beginning of the process, the initial state X_0 (which is 1 or 2, with probabilities 1/3 and 2/3, respectively) is transmitted to a remote site. The received signal is $Y = X_0 + W$, where W is an independent noise term, distributed according to the standard normal distribution (i.e., its mean is 0 and its variance is 1).

(7 pts) **3(f)(i)** Find the probability that the process started in state 1 given that $Y = y$.

$\mathbf{P}$(process started in state 1| $Y = y$)=____________________________________

Show your reasoning and calculations here:

(repeated) **3(f)** At the beginning of the process, the initial state X_0 (which is 1 or 2, with probabilities 1/3 and 2/3, respectively) is transmitted to a remote site. The received signal is $Y = X_0 + W$, where W is an independent noise term, distributed according to the standard normal distribution (i.e., its mean is 0 and its variance is 1).

(7 pts) **3(f)(ii)** Find the conditional PDF of Y^2 given that $X_0 = 1$.

Show your reasoning and calculations here: