Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Quiz 2 Review Solutions
April 8, 2005

1. (a) Since X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} 1 & \text{if } 0 < x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define the event R_1 as $X \leq 0.25$ and the event R_2 as $Y \leq 0.25$. A message is *received* 15 minutes after A sent both messages whenever at least one of R_1 and R_2 occurs. The probability we wish to compute is thus

$$\mathbf{P}(R_1 \cup R_2) = \mathbf{P}(R_1) + \mathbf{P}(R_2) - \mathbf{P}(R_1 \cap R_2).$$

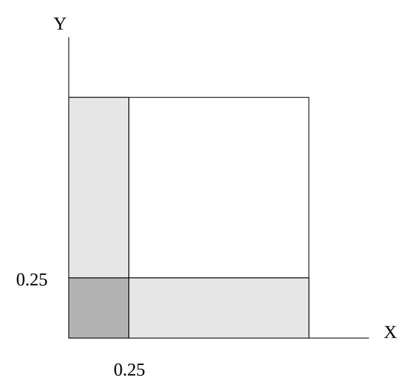
We compute the individual terms as

$$\begin{aligned} \mathbf{P}(R_1) &= \mathbf{P}\left(X \leq \frac{1}{4}\right) = \frac{1}{4}, \\ \mathbf{P}(R_2) &= \mathbf{P}\left(Y \leq \frac{1}{4}\right) = \frac{1}{4}, \\ \mathbf{P}(R_1 \cap R_2) &= \mathbf{P}(R_1) \cdot \mathbf{P}(R_2) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}, \quad \text{since } X \text{ and } Y \text{ are independent.} \end{aligned}$$

Thus the desired probability is

$$\frac{1}{4} + \frac{1}{4} - \frac{1}{16} = \frac{7}{16}.$$

Note also that the probability is the total area of the shaded regions in the following sketch.



(b) Let B be the event that the message is received but not verified within 15 minutes. Then

$$B = R_1 \cap R_2^c \cup R_1^c \cap R_2$$

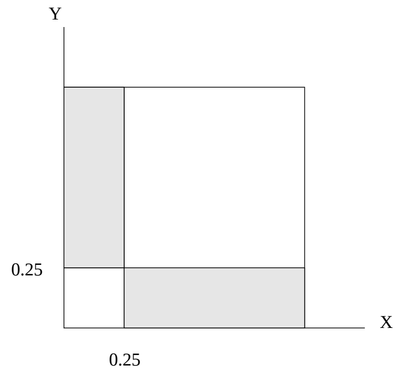
Note that this is a union of disjoint events, so we have

$$\mathbf{P}(B) = \mathbf{P}(R_1 \cap R_2^c) + \mathbf{P}(R_1^c \cap R_2),$$

and the independence of R_1 and R_2 allows the simplification

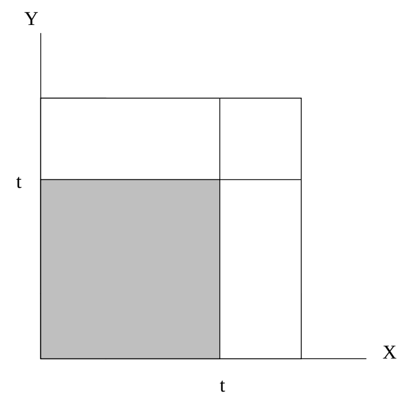
$$\begin{aligned} \mathbf{P}(B) &= \mathbf{P}(R_1) \cdot \mathbf{P}(R_2^c) + \mathbf{P}(R_1^c) \cdot \mathbf{P}(R_2) \\ &= \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{8}. \end{aligned}$$

Note also that the probability is the total area of the shaded regions in the following sketch.



(c) Verification occurs when the second of the messages arrives, so for $t \in [0, 1]$

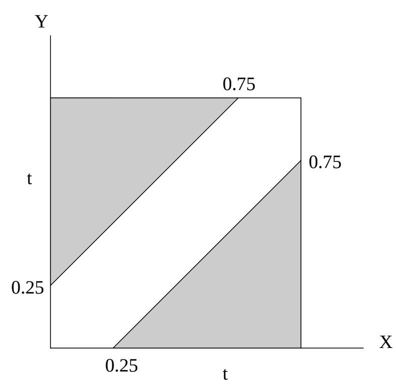
$$\begin{aligned} F_T(t) &= \mathbf{P}(T \le t) \;=\; \mathbf{P}(X \le t \cap Y \le t) \\ &= \mathbf{P}(X \le t) \cdot \mathbf{P}(Y \le t) \quad \text{by the independence of } X \text{ and } Y \\ &= t \cdot t = t^2. \end{aligned}$$



From this we can deduce the full CDF and differentiate to determine the PDF:

$$F_T(t) = \begin{cases} 0 & \text{if } -\infty < t \le 0, \\ t^2 & \text{if } 0 < t \le 1, \\ 1 & \text{if } 1 < t < \infty \end{cases} \qquad \Rightarrow \qquad f_T(t) = \begin{cases} 2t & \text{if } 0 \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(d) The event that the clerk will be there to receive the message is $\{|X - Y| > \frac{1}{4}\}$. We can deduce the probability of this event easily from a sketch:



$$\mathbf{P}(|X - Y| > \frac{1}{4}) = 2 \cdot \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \right) = \frac{9}{16}.$$

(e) We know the strategy from (d) has $\frac{5}{16}$ probability of verification. The other strategy, of sending the employee home after 45 minutes, has probability of verification $\mathbf{P}(T \le \frac{3}{4}) = \frac{5}{16}$ by evaluating the expression from part (c). Therefore, the two strategies are equally effective.

2. First let's write out the properties of all of our random variables. Let us also define K to be the number of members attending a meeting, and B to be the Bernoulli random variable describing whether or not a member attends a meeting.

$$\begin{aligned} M_N(s) &= \frac{(1-p)e^s}{1-pe^s}, \quad \mathbf{E}[N] = \frac{1}{1-p}, \quad \text{var}(N) = \frac{p}{(1-p)^2} \\ M_M(s) &= \frac{\lambda}{\lambda - s}, \quad \mathbf{E}[M] = \frac{1}{\lambda}, \quad \text{var}(M) = \frac{1}{\lambda^2} \\ M_B(s) &= 1 - q - qe^s, \quad \mathbf{E}[B] = q, \quad \text{var}(B) = q(1-q) \end{aligned}$$

(a) Since B and N are independent,

$$\mathbf{E}[K] = \mathbf{E}[N] \cdot \mathbf{E}[B] = \frac{q}{1-p}, \quad \text{var}(K) = \mathbf{E}[N] \cdot {\sigma_B}^2 + {\mu_B}^2 \cdot \text{var}(N) = \frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}$$

(b) We begin first by finding $M_K(s)$ and then we evaluate $p_K(1)$ using properties of the transform. To do this, we recognize that $K = B_1 + B_2 + B_3 + ... + B_N$.

$$\begin{aligned} M_K(s) &= M_N(s)\Big|_{e^s = M_B(s)} = \frac{(1-p)(1-q+qe^s)}{1-p(1-q+qe^s)} \\ p_K(1) &= \frac{d}{de^s} M_K(s)\Big|_{e^s=0} = \left[\frac{(1-p)q}{1-p(1-q+qe^s)} + \frac{pq(1-p)(1-q+qe^s)}{[1-p(1-q+qe^s)]^2} \right]\Bigg|_{e^s=0} \\ &= \frac{(1-p)q}{1-p(1-q)} + \frac{pq(1-p)(1-q)}{[1-p(1-q)]^2} \end{aligned}$$

(c) Let G be the total money brought to the meeting. Then $G = M_1 + M_2 + M_3 + ... + M_K$. Thus we can write the transform of G as follows:

$$M_G(s) = M_K(s)\Big|_{e^s = M_M(s)} = \frac{(1-p)\left[1 - q + q\left(\frac{\lambda}{\lambda - s}\right)\right]}{1 - p\left[1 - q + q\left(\frac{\lambda}{\lambda - s}\right)\right]}$$

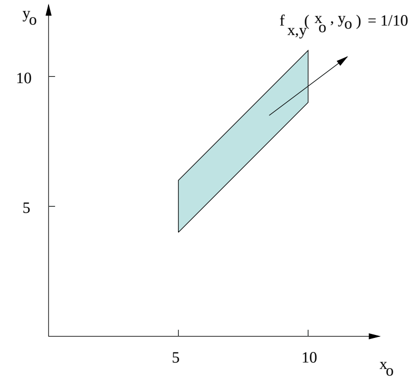
3. (a) The minimum mean squared error estimator $g(Y)$ is known to be $g(Y) = \mathbf{E}[X|Y]$. Let us first find $f_{X,Y}(x,y)$. Since $Y = X + W$, we can write

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2} & \text{if } x - 1 \le y \le x + 1, \\ 0 & \text{otherwise,} \end{cases}$$

and, therefore,

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = \begin{cases} \frac{1}{10} & \text{if } x - 1 \le y \le x + 1 \text{ and } 5 \le x \le 10, \\ 0 & \text{otherwise,} \end{cases}$$

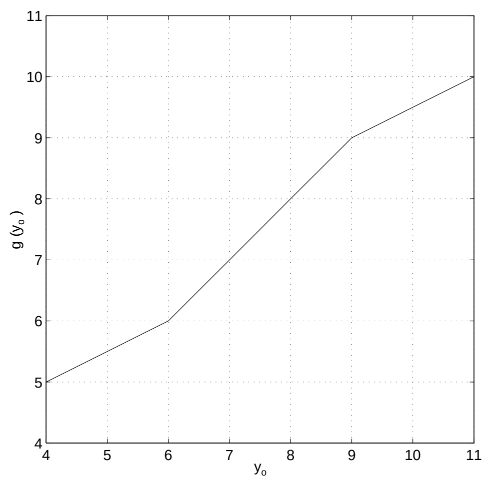
as shown in the plot below.



We now compute $\mathbf{E}[X|Y]$ by first determining $f_{X|Y}(x|y)$. This can be done by looking at the horizontal line crossing the compound PDF. Since $f_{X,Y}(x,y)$ is uniformly distributed in the defined region, $f_{X|Y}(x|y)$ is uniformly distributed as well. Therefore,

$$g(y) = \mathbf{E}[X|Y=y] = \begin{cases} \frac{5+(y+1)}{2} & \text{if } 4 \le y < 6, \\ y & \text{if } 6 \le y \le 9, \\ \frac{10+(y-1)}{2} & \text{if } 9 < y \le 11. \end{cases}$$

The plot of $g(y)$ is shown here.



(b) The linear least squares estimator has the form

$$g_L(Y) = \mathbf{E}[X] + \frac{\text{cov}(X,Y)}{\sigma_Y^2}(Y - \mathbf{E}[Y])$$

where $\text{cov}(X,Y) = \mathbf{E}[(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])]$. We compute $\mathbf{E}[X] = 7.5$, $\mathbf{E}[Y] = \mathbf{E}[X]+\mathbf{E}[W] = 7.5$, $\sigma_X^2 = (10-5)^2/12 = 25/12$, $\sigma_W^2 = (1-(-1))^2/12 = 4/12$ and, using the fact that X and W are independent, $\sigma_Y^2 = \sigma_X^2 + \sigma_W^2 = 29/12$. Furthermore,

$$\begin{aligned} \text{cov}(X,Y) &= \mathbf{E}[(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])] = \mathbf{E}[(X-\mathbf{E}[X])(X-\mathbf{E}[X]+W-\mathbf{E}[W])] \\ &= \mathbf{E}[(X-\mathbf{E}[X])(X-\mathbf{E}[X])] + \mathbf{E}[(X-\mathbf{E}[X])(W-\mathbf{E}[W])] \\ &= \sigma_X^2 + \mathbf{E}[(X-\mathbf{E}[X])]\mathbf{E}[(W-\mathbf{E}[W])] = \sigma_X^2 = 25/12. \end{aligned}$$

Note that we use the fact that $(X-\mathbf{E}[X])$ and $(W-\mathbf{E}[W])$ are independent and $\mathbf{E}[(X-\mathbf{E}[X])] = 0 = \mathbf{E}[(W-\mathbf{E}[W])]$. Therefore,

$$g_L(Y) = 7.5 + \frac{25}{29}(Y - 7.5).$$

The linear estimator $g_L(Y)$ is compared with $g(Y)$ in the following figure. Note that $g(Y)$ is piecewise linear in this problem.