MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Recitation 11 Solutions
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1. Note that the moment generating functions for X, Y, and Z are

$$\begin{aligned} M_X(s) &= \frac{3}{4} + \frac{1}{4}e^s, \\ M_Y(s) &= \frac{3}{3-s}, \text{ for } s < 3, \text{ and} \\ M_Z(s) &= e^{5(e^s-1)}. \end{aligned}$$

(a) By direct substitution of $5Z+1$ in the expectation,

$$M_{5Z+1}(s) = \mathbf{E}[e^{s(5Z+1)}] = e^s\mathbf{E}[s^{s(5Z)}] = e^s M_Z(5s) = e^s e^{5(e^{5s}-1)},$$

which could also have been obtained by the formula provided in Chapter 4 of the text.

(b) Since X and Y are independent,

$$M_{X+Y}(s) = M_X(s)M_Y(s) = \left(\frac{3}{4} + \frac{1}{4}e^s\right)\frac{3}{3-s}, \text{for } s < 3.$$

(c) We can use the total expectation theorem to find the transform of U.

$$\begin{aligned} M_U(s) &= \mathbf{P}(X=1)\mathbf{E}[e^{sU}|X=1] + \mathbf{P}(X=0)\mathbf{E}[e^{sU}|X=0] \\ &= \mathbf{P}(X=1)\mathbf{E}[e^{s(1\cdot Y+0\cdot Z)}|X=1] + \mathbf{P}(X=0)\mathbf{E}[e^{s(0\cdot Y+1\cdot Z)}|X=0] \\ &= \mathbf{P}(X=1)\mathbf{E}[e^{sY}|X=1] + \mathbf{P}(X=0)\mathbf{E}[e^{sZ}|X=0] \end{aligned}$$

But X and Y are independent so

$$\mathbf{E}[e^{sY}|X=1] = \mathbf{E}[e^{sY}] = M_Y(s)$$

and

$$\mathbf{E}[e^{sZ}|X=0] = \mathbf{E}[e^{sZ}] = M_Z(s).$$

Therefore,

$$\begin{aligned} M_U(s) &= \frac{1}{4}M_Y(s) + \frac{3}{4}M_Z(s) \\ &= \frac{1}{4}\cdot\frac{3}{3-s} + \frac{3}{4}\cdot e^{5(e^s-1)} \text{ for } s < 3. \end{aligned}$$

2. X is the mixture of two exponential random variables with parameters 1 and 3, which are selected with probability 1/3 and 2/3, respectively. Hence, the PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{3}\cdot e^{-x} + \frac{2}{3}\cdot 3e^{-3x} & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. The coefficients a, b, and c are determined from the three relations

$$M(0) = 1, \quad \frac{d}{ds}M(s)\Big|_{s=0} = \mathbf{E}[X] = 3, \quad \frac{d^2}{ds^2}M(s)\Big|_{s=0} = \text{var}(X) + \big(\mathbf{E}[X]\big)^2 = 11.$$

These equations are written as

$$a + b + c = 1, \quad 2b + 4c = 3, \quad 4b + 16c = 11.$$

Solving this system of equations, we obtain $a = 1/8$, $b = 2/8$, $c = 5/8$.

The PMF of X can be deduced from the powers of e^s and their coefficients in the expression for $M(s)$. We have

$$\mathbf{P}(X = 0) = a = 1/8, \qquad \mathbf{P}(X = 2) = b = 2/8, \qquad \mathbf{P}(X = 4) = c = 5/8.$$