

6.041 Spring 2005 Final Exam  
Tuesday, May 17, 1:30–4:30 p.m.

DO NOT TURN THIS EXAM OVER UNTIL  
YOU ARE TOLD TO DO SO

- You have 3 hours to complete the exam.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This exam has three problems, each with multiple parts, that are not necessarily in order of difficulty.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like  $\binom{8}{3}$  or  $\sum_{k=0}^5 (1/2)^k$  are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for three handwritten, 2-sided 8.5x11 formula sheets plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the exam, turn in your solutions along with this exam (this piece of paper).

**Write your name, your recitation instructor's name, and your TA's name on the front of the booklet.** (3 points)

**Problem 1.** (26 points) Let  $X_1, X_2, \dots$  be an i.i.d. sequence of normal random variables with mean  $\mu$  and variance  $\sigma^2$ . Further, let

$$Y_n = \frac{\sum_{i=1}^n X_i}{\sqrt{n}}$$

for  $n = 1, 2, \dots$

- (a) (6 points) Using the Chebyshev inequality, give an upper bound for  $\mathbf{P}(|Y_n - \mathbf{E}[Y_n]| \geq \epsilon)$ .
- (b) (7 points) Evaluate  $\mathbf{P}(|Y_n - \mathbf{E}[Y_n]| \geq \epsilon)$  exactly in terms of  $\Phi$ , the CDF of the standard normal random variable.
- (c) (6 points) Hence compute  $\mathbf{P}(|Y_n - \mathbf{E}[Y_n]| \geq \epsilon)$  and its Chebyshev upper bound for  $\epsilon/\sigma = 0.5$ ,  $\epsilon/\sigma = 1.0$ , and  $\epsilon/\sigma = 2.0$ .
- (d) (7 points) For  $n > k$ , find the linear least squares estimate of  $Y_n$  given  $Y_k = y$  and its mean squared error.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Electrical Engineering & Computer Science  
**6.041/6.431: Probabilistic Systems Analysis**  
(Spring 2005)

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**Problem 2.** (36 points) Two kinds of vehicles go to Harvard Square from 77 Mass. Ave.: taxis and buses. The interarrival time of taxis, in minutes, is an independent exponential random variable with parameter  $\lambda_1$ , i.e. its PDF is  $f_{I_T}(t) = \lambda_1 e^{-\lambda_1 t}$  for  $t \geq 0$ , while the interarrival time of buses, in minutes, is an independent exponential random variable with parameter  $\lambda_2$ , i.e., its PDF is  $f_{I_B}(t) = \lambda_2 e^{-\lambda_2 t}$  for  $t \geq 0$ .

Suppose Joe and Harry arrive at 77 Mass. Ave. at 7:00 a.m.

- (a) (5 points) What is the average time before they see the first vehicle?
- (b) (5 points) What is the probability that the first vehicle they see is a bus, and what is the probability that the first vehicle they see is a taxi?

In a taxi, the travel time to Harvard Square, in minutes, is an independent exponential random variable with parameter  $\mu_1$ , i.e., its PDF is  $f_{D_T}(t) = \mu_1 e^{-\mu_1 t}$  for  $t \geq 0$ . On the other hand, in a bus, the travel time to Harvard Square, in minutes, is an independent exponential random variable with parameter  $\mu_2$ , i.e., its PDF is  $f_{D_B}(t) = \mu_2 e^{-\mu_2 t}$  for  $t \geq 0$ .

- (c) (7 points) Suppose Joe and Harry arrive at 77 Mass. Ave. at 7:00 a.m., take the first vehicle that passes, and arrive at Harvard Square  $X$  minutes later. Find the transform of  $X$ .
- (d) (7 points) Suppose that a taxi and a bus arrive simultaneously, and Joe takes the taxi while Harry takes the bus. Let  $Y$  be the number of minutes from their departure from 77 Mass. Ave. till they meet again at Harvard Square. Find  $\mathbf{E}[Y]$ .

There are, in fact, two different kinds of buses: fast buses and slow buses. For any bus that arrives at 77 Mass. Ave., it is a fast bus with probability  $p$ , and it is a slow bus with probability  $1 - p$ . Whether a bus is fast or slow is independent of everything else.

- (e) (5 points) If they stay at 77 Mass. Ave. for  $l$  minutes, how many fast buses will they see on average?
- (f) (7 points) If they stay indefinitely, what is the probability that they will see  $k$  fast buses before they see  $k$  slow buses?

**Problem 3.** (35 points) A hungry mouse is trapped in a cage with three doors. At each “turn”, the mouse gets to open one of the three doors and eat a piece of cheese behind the door if one is present. Each door is chosen with equal probability on each turn, regardless of whether a piece of cheese was found on the previous turn. If no cheese was found on the previous turn, there is a probability of  $3/4$  that cheese will be found behind each door on the current turn. If cheese was found on the previous turn and the same door is chosen on the current turn, then there is a probability of 0 that cheese will be found; whilst if cheese was found on the previous turn and a different door is chosen on the current turn, then there is a probability of 1 that cheese will be found.

- (a) (7 points) If you observe the mouse’s behavior over 1000 turns, in approximately what fraction of turns do you expect the mouse to eat a piece of cheese?
- (b) (7 points) Suppose no cheese was found on the previous turn. What is the expected number of turns before the mouse eats a piece of cheese?
- (c) (7 points) Suppose no cheese was found on the previous turn. What is the expected number of turns before the mouse eats  $n$  pieces of cheese?
- (d) (7 points) Suppose cheese was found on the previous turn. Using the Central Limit Theorem, approximate the probability that the number of turns before the mouse eats 100 pieces of cheese exceeds 152.
- (e) (7 points) You look into the cage and observe the mouse eating a piece of cheese from behind door number 1. What is the probability that, if you observe the mouse three turns later, it will again be eating a piece of cheese from behind door number 1?