Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Quiz 1 Review Solutions
March 4, 2005

1. Consider a sequence of six independent rolls of this die, and let x_i be the random variable corresponding to the ith roll.

   (a) What is the probability that exactly three of the rolls have an outcome equal to three? Each roll x_i can either be a three with probability 1/4 or not a three with probability 3/4. There are $\binom{6}{3}$ ways of placing the threes in the sequence of 6 rolls. After we require that a three go in each of these spots, which takes probablity $\frac{1}{4}^3$ our only remaining condition is that either a one or a two go in the other three spots, which takes probability $\frac{3}{4}^3$. So the probability of exactly three rolls in a sequence of 6 independent rolls is $\boxed{\binom{6}{3}(\frac{1}{4})^3(\frac{3}{4})^3}$.

   (b) What is the probability that the first roll is a 1, given that exactly two of the six rolls had an outcome of 1? The probability of obtaining a one on a single roll is 1/2, and the probability of obtaining a 2 or 3 on a single roll is also 1/2. For the purposes of solving this problem we treat obtaining a 2 or 3 as an equivalent outcome. We know that there are $\binom{6}{2}$ ways of rolling exactly 2 ones. Of these $\binom{6}{2}$ ways exactly $\binom{5}{1} = 5$ ways result in a one in the first roll, since we can place the remaining one in any of the 5 remaining rolls. The rest of the rolls must be either two or three. Thus the probability that the first roll is a one given exactly 2 rolls had an outcome of one is $\boxed{\frac{5}{\binom{6}{2}}}$.

   (c) We are now told that exactly three of the rolls resulted in one and exactly three resulted in 2. What is the probability of the outcome 121212? We want to find

   $$\mathbf{P}(121212 \,|\, \text{exactly 3 ones and 3 twos}) = \frac{\mathbf{P}(121212)}{\mathbf{P}(\text{exactly 3 ones and 3 twos})}.$$

   Any particular sequence of three ones and three twos will have the same probability: $\frac{1}{2}^3\frac{1}{4}^3$. There are $\binom{6}{3}$ possible rolls with exactly three ones and three twos.
   Therefore $\mathbf{P}(121212 \,|\, \text{exactly 3 ones and 3 twos}) = \boxed{\frac{1}{\binom{6}{3}}}$.

   (d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's. Let A be the event that at least one roll results in a three. Then $\mathbf{P}(A) = 1 - \mathbf{P}(\text{no rolls resulted in three}) = 1 - (\frac{3}{4})^6$. Now let k be the random variable representing the number of threes in the 6 rolls. Our unconditional PMF $p_K(k)$ for k is given by

   $$p_K(k) = \binom{6}{k}\frac{1}{4}^k\frac{3}{4}^{6-k}.$$

   We find the conditional PMF $p_{k|A}(k|A)$ for k using the definition of conditional probability:

   $$p_{K|A}(k|A) = \frac{\mathbf{P}(K=k, A)}{\mathbf{P}(A)}.$$

   Thus we obtain

   $$p_{K|A}(k|A) = \begin{cases} \frac{1}{1-(3/4)^6}\binom{6}{k}(\frac{1}{4})^k(\frac{3}{4})^{6-k} & \text{for } k = 1, 2, \ldots, 6, \\ 0 & \text{otherwise.} \end{cases}$$

   Note that $p_{K|A}(0|A) = 0$ because the event $k = 0$ and the event A are mutually exclusive. Thus the probability of their intersection, which appears in the numerator in the definition of the conditional PMF, is zero.

2. Suppose the president decides to investigate A first. Then her expected costs will be

$$\mathbf{E}[\text{costs}] = D_A + pR_A + (1-p)\cdot(D_B + R_B),$$

whereas if she investigates B first, then

$$\mathbf{E}[\text{costs}] = D_B + (1-p)\cdot R_B + p\cdot(D_A + R_A).$$

In order that the first be smaller than the second, we need

$$pD_B > (1-p)D_A.$$

3. (a) Use the total probability theorem by conditioning on the number of questions that May has to answer. Let A be the event that she gives all wrong answers in a given lecture, let B_1 be the event that she gets one question in a given lecture, and let B_2 be the event that she gets two questions in a given lecture. Then

$$\mathbf{P}(A) = \mathbf{P}(A|B_1)\mathbf{P}(B_1) + \mathbf{P}(A|B_2)\mathbf{P}(B_2).$$

From the problem statement, she is equally likely to get one or two questions in a given lecture, so $\mathbf{P}(B_1) = \mathbf{P}(B_2) = \frac{1}{2}$. Also, from the problem, $\mathbf{P}(A|B_1) = \frac{1}{4}$, and, because of independence, $\mathbf{P}(A|B_2) = (\frac{1}{4})^2 = \frac{1}{16}$. Thus we have

$$\mathbf{P}(A) = \frac{1}{4}\frac{1}{2} + \frac{1}{16}\frac{1}{2} = \frac{5}{32}.$$

(b) Let events A and B_2 be defined as in the previous part. Using Bayes's Rule:

$$\mathbf{P}(B_2|A) = \frac{\mathbf{P}(A|B_2)\mathbf{P}(B_2)}{\mathbf{P}(A)}.$$

From the previous part, we said $\mathbf{P}(B_2) = \frac{1}{2}$, $\mathbf{P}(A|B_2) = \frac{1}{16}$, and $\mathbf{P}(A) = \frac{5}{32}$. Thus

$$\mathbf{P}(B_2|A) = \frac{\frac{1}{16}\frac{1}{2}}{\frac{5}{32}} = \frac{1}{5}.$$

As one would expect, given that May answers all the questions in a given lecture, it's more likely that she got only one question rather than two.

(c) We start by finding the PMF for X and Y. $p_X(x)$ is given from the problem statement:

$$p_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in \{1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

The PMF for Y can be found by conditioning on X for each value that Y can take on. Because May can be asked at most two questions in any lecture, the range of Y is from 0 to 2. Thus for each value of Y, we find

$$p_Y(0) = \mathbf{P}(Y=0|X=0)\mathbf{P}(X=1) + \mathbf{P}(Y=0|X=2)\mathbf{P}(X=2) = \frac{1}{4}\frac{1}{2} + \frac{1}{16}\frac{1}{2} = \frac{5}{32},$$

$$p_Y(1) = \mathbf{P}(Y=1|X=1)\mathbf{P}(X=1) + \mathbf{P}(Y=1|X=2)\mathbf{P}(X=2) = \frac{3}{4}\frac{1}{2} + 2\frac{3}{4}\frac{1}{4}\frac{1}{2} = \frac{9}{16},$$

$$p_Y(2) = \mathbf{P}(Y=2|X=1)\mathbf{P}(X=1) + \mathbf{P}(Y=2|X=2)\mathbf{P}(X=2) = 0\frac{1}{2} + (\frac{3}{4})^2\frac{1}{2} = \frac{9}{32}.$$

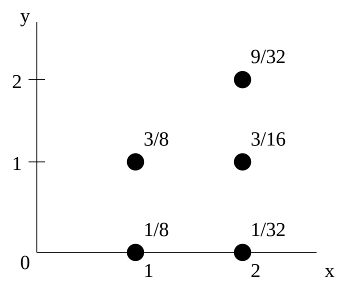
Note that when calculating $\mathbf{P}(Y = 1|X = 2)$, we got $2\frac{3}{4}\frac{1}{4}$ because there are two ways for May to answer one question right when she's asked two questions: either she answers the first question correctly or she answers the second question correctly. Thus, overall

$$p_Y(y) = \begin{cases} 5/32 & \text{if } y = 0, \\ 9/16 & \text{if } y = 1, \\ 9/32 & \text{if } y = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Now the mean and variance can be calculated explicitly from the PMFs:

$$\mathbf{E}[X] = 1\frac{1}{2} + 2\frac{1}{2} = \frac{3}{2},$$
$$\text{var}(X) = (1 - \frac{3}{2})^2\frac{1}{2} + (2 - \frac{3}{2})^2\frac{1}{2} = \frac{1}{4},$$
$$\mathbf{E}[Y] = 0\frac{5}{32} + 1\frac{9}{16} + 2\frac{9}{32} = \frac{9}{8},$$
$$\text{var}(Y) = (0 - \frac{9}{8})^2\frac{5}{32} + (1 - \frac{9}{8})^2\frac{9}{16} + (2 - \frac{9}{8})^2\frac{9}{32} = \frac{27}{64}.$$

(d) The joint PMF $p_{X,Y}(x, y)$ is plotted below. There are only five possible (x, y) pairs. For each point, $p_{X,Y}(x, y)$ was calculated by $p_{X,Y}(x, y) = p_X(x)p_{Y|X}(y|x)$.



(e) By linearity of expectations,

$$\mathbf{E}[Z] = \mathbf{E}[X + 2Y] = \mathbf{E}[X] + 2\mathbf{E}[Y] = \frac{3}{2} + 2\frac{9}{8} = \frac{15}{4}.$$

Calculating var(Z) is a little bit more tricky because X and Y are not independent; therefore we *cannot* add the variance of X to the variance of $2Y$ to obtain the variance of Z. (X and Y are clearly not independent because if we are told, for example, that $X = 1$, then we know that Y cannot equal 2, although normally without any information about X, Y could equal 2.)

To calculate var(Z), first calculate the PMF for Z from the joint PDF for X and Y. For each (x, y) pair, we assign a value of Z. Then for each value z of Z, we calculate $p_Z(z)$ by summing over the probabilities of all (x, y) pairs that map to z. Thus we get

$$p_Z(z) = \begin{cases} 1/8 & \text{if } z = 1, \\ 1/32 & \text{if } z = 2, \\ 3/8 & \text{if } z = 3, \\ 3/16 & \text{if } z = 4, \\ 9/32 & \text{if } z = 6, \\ 0 & \text{otherwise.} \end{cases}$$

In this example, each (x, y) mapped to exactly one value of Z, but this does not have to be the case in general. Now the variance can be calculated as:

$$\text{var}(Z) = \frac{1}{8}(1 - \frac{15}{4})^2 + \frac{1}{32}(2 - \frac{15}{4})^2 + \frac{3}{8}(3 - \frac{15}{4})^2 + \frac{3}{16}(4 - \frac{15}{4}) + \frac{9}{32}(6 - \frac{15}{4})^2 = \frac{43}{16}.$$

(f) For each lecture i, let Z_i be the random variable associated with the number of questions May gets asked plus two times the number May gets right. Also, for each lecture i, let D_i be the random variable $1000 + 40Z_i$. Let S be her semesterly salary. Because she teaches a total of 20 lectures, we have

$$S = \sum_{i=1}^{20} D_i = \sum_{i=1}^{20} 1000 + 40Z_i = 20000 + 40 \sum_{i=1}^{20} Z_i.$$

By linearity of expectations,

$$\mathbf{E}[S] = 20000 + 40\mathbf{E}[\sum_{i=1}^{20} Z_i] = 20000 + 40(20)\mathbf{E}[Z_i] = 23000.$$

Since each of the D_i are independent, we have

$$\text{var}(S) = \sum_{i=1}^{20} \text{var}(D_i) = 20\text{var}(D_i) = 20\text{var}(1000 + 40Z_i) = 20(40^2\text{var}(Z_i)) = 36000.$$

(g) Let Y be the number of questions she will answer wrong in a randomly chosen lecture. We can find $\mathbf{E}[Y]$ by conditioning on whether the lecture is in math or in science. Let M be the event that the lecture is in math, and let S be the event that the lecture is in science. Then

$$\mathbf{E}[Y] = \mathbf{E}[Y|M]\mathbf{P}(M) + \mathbf{E}[Y|S]\mathbf{P}(S).$$

Since there are an equal number of math and science lectures and we are choosing randomly among them, $\mathbf{P}(M) = \mathbf{P}(S) = \frac{1}{2}$. Now we need to calculate $\mathbf{E}[Y|M]$ and $\mathbf{E}[Y|S]$ by finding the respective conditional PMFs first. The PMFs can be determined in an manner analagous to how we calculated the PMF for the number of correct answers in part (c).

$$p_{Y|S}(y) = \begin{cases} \frac{1}{2}\frac{3}{4} + \frac{1}{2}(\frac{3}{4})^2 = 21/32 & \text{if } y = 0, \\ \frac{1}{2}\frac{1}{4} + \frac{1}{2}2\frac{1}{4}\frac{3}{4} = 5/16 & \text{if } y = 1, \\ \frac{1}{2}0 + \frac{1}{2}(\frac{1}{4})^2 = 1/32 & \text{if } y = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$p_{Y|M}(y) = \begin{cases} \frac{1}{2}\frac{9}{10} + \frac{1}{2}(\frac{9}{10})^2 = 171/200 & \text{if } y = 0, \\ \frac{1}{2}\frac{1}{10} + \frac{1}{2}2\frac{1}{10}\frac{9}{10} = 7/50 & \text{if } y = 1, \\ \frac{1}{2}0 + \frac{1}{2}(\frac{1}{10})^2 = 1/200 & \text{if } y = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\mathbf{E}[Y|S] = 0\frac{21}{32} + 1\frac{5}{16} + 2\frac{1}{32} = \frac{3}{8},$$

$$\mathbf{E}[Y|M] = 0\frac{171}{200} + 1\frac{7}{50} + 2\frac{1}{200} = \frac{3}{20}.$$

This implies that

$$\mathbf{E}[Y] = \frac{3}{20}\frac{1}{2} + \frac{3}{8}\frac{1}{2} = \frac{21}{80}.$$