Tutorial 8 Solutions
Week of April 4, 2005

1. Let X = time between successive bites.
   Let G = time until the next bite.
   We have $X = G - 1$.

   The mosquito bites occur according to a Bernoulli Process with parameter $p = 0.1$. G is a geometric random variable, so, $\mathbf{E}[G] = \frac{1}{p} = \frac{1}{0.1} = 10$. Therefore, $\mathbf{E}[X] = \mathbf{E}[G-1] = 9$.

$$\text{Var}(X) = \text{Var}(G-1) = \text{Var}(G) = \frac{1-p}{p^2} = \frac{1-0.1}{0.1^2} = 90.$$

   .

2. (a) If a bulb fails, the probability that it was a bulb of Type-A is the same as the probability of having chosen a bulb of type A, which is 1/2. The number of Type-A bulb failures among the first 11 bulb failures is a binomial r.v. with parameters $(11, 1/2)$. Once we know there were 3 Type-A bulb failures among the first 11 failures (which according to the binomial distribution, occurs with probability $\binom{11}{3}(\frac{1}{2})^{11}$), the probability that the 12^{th} failure is a Type-A bulb failure is again 1/2. Therefore

$$\mathbf{P}(E) = \binom{11}{3}(\frac{1}{2})^{12}.$$

   (b) The number of Type-A bulb failures among the first 12 bulb failures is a binomial r.v. with parameters $(12, 1/2)$. Event F corresponds to the event that this r.v. takes the value 4, so using the binomial PMF,

$$\mathbf{P}(F) = \binom{12}{4}(\frac{1}{2})^{12}$$

   (c) U is sum of 12 independent bulb failure times, i.e. $U = Y_1 + Y_2 + ... + Y_{12}$ with Y_i : time until the i^{th} failure.
   We notice that $Y_i,\ i = 1, \ldots, 12$ are i.i.d. therefore

$$M_U(s) = \big(M_{Y_1}(s)\big)^{12}.$$

   With probability 1/2, Y_1 is the time until failure of a Type-A bulb, and with probability 1/2 it is the time until failure of a Type-B bulb. As a result,

$$f_{Y_1}(x) = (1/2)e^{-x} + (1/2)3e^{-3x}, \quad x \geq 0$$

   and

$$M_{Y_1}(s) = \frac{1}{2}\frac{1}{1-s} + \frac{1}{2}\frac{3}{3-s}.$$

   Therefore

$$M_U(s) = \left[\frac{1}{2}\left(\frac{1}{1-s} + \frac{3}{3-s}\right)\right]^{12}$$

(d) First, notice that T the total lifetime of the first two Type-B bulbs is a second order erlang with rate 3:

$$f_T(t) = 3^2 te^{-3t}, \ t \geq 0.$$

To find the probablity that T is greater than the lifetime of one Type-A bulb, we first determine the joint density function of A and T.
Because of independence,

$$f_{A,T}(a,t) \ = \ f_A(a) * f_T(t) \ = \ 9te^{-3t}e^{-a}, \quad a,t \geq 0.$$

Then $\mathbf{P}(G)$ can be obtained by taking the double integral of the joint density function over the region where the value of A is lower than the value of T:

$$\mathbf{P}(G) = \mathbf{P}(T > A) = \int_0^\infty \int_0^t 9te^{-3t}e^{-a}da \ dt = \frac{7}{16}.$$

(e) $V = B_1 + B_2 + ... + B_K$ where B_i is the lifetime of the i^{th} Type-B bulb, and K is the number of Type-B bulbs in the first twelve bulbs used. Notice that K is a random variable that has a binomial distribution with parameters $(12, 1/2)$ and that $B_i, \ i = 1, 2, \ldots$ are i.i.d. with an exponential distribution with parameter 3.
Using the law of iterated expectations,

$$\mathbf{E}[V] = \mathbf{E}[\mathbf{E}[V|K]] = \mathbf{E}[K\mathbf{E}[B]] = \mathbf{E}[B]\mathbf{E}[K] = \frac{1}{3}(\frac{1}{2}.12) = 2.$$

Using the law of conditional variances,

$$\begin{aligned}
\text{Var}(V) &= \text{Var}(\mathbf{E}[V|K]) + \mathbf{E}[\text{Var}(V|K)] \\
&= \text{Var}(K\mathbf{E}[B]) + \mathbf{E}[K\text{Var}(B)] \\
&= (\mathbf{E}[B])^2\text{Var}(K) + \text{Var}(B)\mathbf{E}[K] \\
&= (\frac{1}{3})^2(\frac{1}{2}(1 - \frac{1}{2})12) + (\frac{1}{3})^2(\frac{1}{2}.12) \\
&= 1
\end{aligned}$$