MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

**6.041/6.431: Probabilistic Systems Analysis**

(Spring 2005)


**Recitation 8: Answers**

**March 8, 2005**

1. (a) No

   (b) $c = \frac{1}{\pi r^2}$.

   (c) $f_X(x) = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & , \ |x| \le r \\ 0 & , \ \text{otherwise} \end{cases}$

   (d) $f_{X|Y}(x \mid y) = \begin{cases} \frac{1}{2\sqrt{r^2-y^2}} & , \ x^2+y^2 \le r^2 \\ \text{undefined} & , \ |y| > r \\ 0 & , \ \text{otherwise} \end{cases}$

   $\mathbf{E}[X \mid Y = \sqrt{3}r/2] = 0$

   (e) $f_{X,Y|A}(x, y \mid A) = \begin{cases} \frac{4}{\pi r^2} & , \ x^2+y^2 \le (r/2)^2 \\ 0 & , \ \text{otherwise} \end{cases}$

2. a)

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) \\ &= \mathbf{P}((X_1 \le y) \cap \cdots \cap (X_n \le y)) \\ &= (\mathbf{P}(X_i \le y))^n \\ &= (y)^n \end{aligned}$$

   b)

$$\begin{aligned} F_W(w) &= \mathbf{P}(W \le w) \\ &= 1 - \mathbf{P}(W \ge w) \\ &= 1 - (\mathbf{P}(X_i \ge w))^n \\ &= 1 - (1-w)^n \end{aligned}$$

   c)

$$\begin{aligned} F_{Y,W}(y, w) &= \mathbf{P}(Y \le y, W \le w) \\ &= \mathbf{P}(W \le w | Y \le y) \cdot P(Y \le y) \\ &= \begin{cases} y^n - (y-w)^n & , \ 0 \le w \le y \le 1 \\ 1 - (1-w)^n & , \ y > 1, 0 \le w \le 1 \\ y^n & , \ w > 1, 0 \le y \le 1 \\ 1 & , \ w > 1, y > 1 \\ 0 & , \ \text{otherwise} \end{cases} \end{aligned}$$

3. Symmetry is a powerful argument, and most importantly, gets us the answer to (a) and (b).

   (a) 1/2

   (b) 1/3

(c) $\mathbf{P}(N = n) = \mathbf{P}(X_1$ is the smallest of 1st $n-1$, *and* $X_n$ is the smallest of first $n)$. So using the extension of the argument in (a) and (b), we have $\mathbf{P}(n = n_0) = \frac{1}{n-1} \times \frac{1}{n}$. Now,

$$\mathbf{P}(N > n) = 1 - \mathbf{P}(N \leq n) = 1 - \sum_{k=2}^{n} \frac{1}{k(k-1)}$$

Alternatively, $\mathbf{P}(N > n) = P(X_1$ is the smallest of 1st $n) = \frac{1}{n}$.