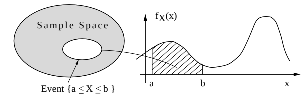
LECTURE 8

- **Readings:** Sections 3.1-3.3

Lecture outline

- Probability density functions
- Cumulative distribution functions
- Normal random variables

Continuous r.v.'s and pdf's

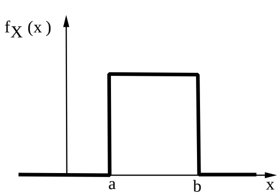


$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

- $\mathbf{P}(x \le X \le x+\delta) \approx f_X(x) \cdot \delta$
- $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Means and variances

- $\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$
- $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$
- var$(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\,dx$
- **Uniform** example:

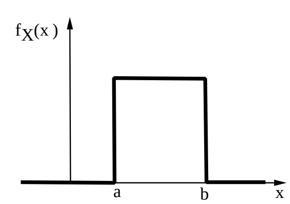


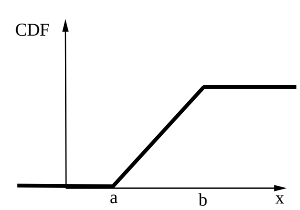
- $f_X(x) = \qquad\qquad a \le x \le b$
- $\mathbf{E}[X] =$
- $\sigma_X^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx = \frac{(b-a)^2}{12}$

Cumulative distribution function (CDF)

- CDF:

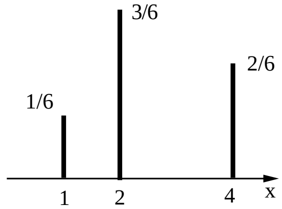
$$F_X(x) = \mathbf{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$$

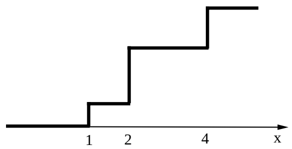




- Also for discrete r.v.'s:

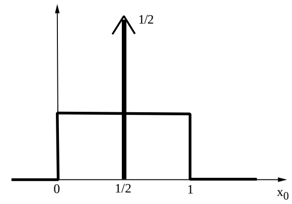
$$F_X(x) = \mathbf{P}(X \le x) = \sum_{k \le x} p_X(k)$$





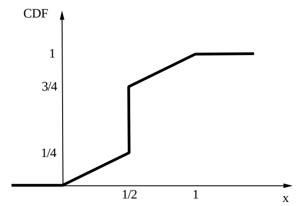
Mixed distributions

- Schematic drawing of a combination of a pdf and a pmf



- The corresponding CDF:

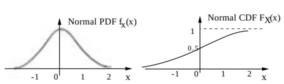
$$F_X(x) = \mathbf{P}(X \le x)$$



Gaussian (normal) PDF

- Standard normal $N(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



- $\mathbf{E}[X] =$ $\text{var}(X) = 1$

- General normal $N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- It turns out that:
  $\mathbf{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

- Let $Y = aX + b$
  - Then, $\mathbf{E}[Y] =$
    $\text{Var}(Y) =$
  - Fact: $Y \sim N(a\mu + b, a^2\sigma^2)$

Calculating normal probabilities

- No closed form available for CDF
  - but there are tables
    (for standard normal)

- If $X \sim N(\mu, \sigma^2)$, then $\dfrac{X - \mu}{\sigma} \sim N($ $)$

- If $X \sim N(2, 16)$:

$$\mathbf{P}(X \le 3) = \mathbf{P}\left(\frac{X-2}{4} \le \frac{3-2}{4}\right) = \text{CDF}(0.25)$$