Problem Set 10
Due: May 4, 2005

1. The MIT football team's performance in any given game is very much correlated to its morale. In fact, if the team has won the past two games, then it has a .7 probability of winning the next game. If it lost the last game but won before that, it has a .4 probability of winning. If it won its last game but lost before that it has a .5 probability of winning, and finally if it lost the last two games it has only a .2 probability of winning the next game. Assume that the above details the complete correlation between the history of victories and defeats, and the future performance.

   (a) Define with a Markov chain that models the above process. Remember that the chain must have the Markov property.

   (b) Find the long run probability that the MIT football team will win its next game.

2. At the Probability Coffee House of MIT, there is only one cashier. Due to the limited space, she allows only M customers to line before her at any time. If a customer finds there are M customers there including the one being served by the cashier, he will leave the Coffee House immdediately.

   Every minute, exactly one of the following occurs:

   - one new customer arrives with probability p;
   - one existing customer leaves with probality kq, where k is the number of customers in the House; or
   - no new customer arrives and no existing customer leaves with probability $1 - p - kq$ if there is at least one customer in the House, and with probability $1 - p$ otherwise.

   (a) This problem can be modeled as a birth-death process. Define appropriate states and draw the transition probability graph.

   (b) After the House has been open for a long time, you walk into the House. Calculate how many customers you expect to see in line.

3. Mr. Mean Variance has the only key which locks or unlocks the door to Building 59, the Probability Building. He visits the door once each hour on the hour. When he arrives:

   If the door is unlocked, he locks it with probability 0.3.
   If the door is locked, he unlocks it with probability 0.8.

   (a) After he has been on the job several months, is he more likely to lock the door or to unlock it on a randomly selected visit?

   (b) With the process in the steady state, Joe arrived at Building 59 two hours ahead of Harry. What is the probability that each of them found the door in the same condition?

   (c) Given the door was open at the time Mr. Variance was hired, determine the expected value of the number of visits up to and including the one on which he unlocks the door himself for the first time.

   In the following, suppose that every time after he leaves the building, he comes back after T hours, where T is equal to 1 with probability 1/2, and to 2 with probability 1/2.

(d) After he has been on the job several months, an absent-minded visitor, Mr. Drunker, visits Building 59 five minutes before a randomly-selected hour. What is the probability that Mr. Drunker will find the door is unlocked? (Hint: Consider modeling the problem as a new Markov process.)

(e) With the process in the steady state, Joe arrives at Building 59 two hours ahead of Harry. What is the probability that each of them found the door in the same condition?

4. Consider two independent Poisson processes with rates $\lambda_A = 1$ and $\lambda_B = 2$. Suppose that we just had a type B arrival (i.e., an arrival from the second process). Find the expected time we get to observe a type A arrival (i.e., from the first process) such that the preceding arrival was also of type A.