P2 (a) Let's associate a Bernoulli random variable X_k to urn k, where "success" corresponds to extracting a white ball. Then, clearly X_k has the probability mass function:

$$X_k = \begin{cases} 1 & \text{with probability } \frac{k}{N+1} \\ 0 & \text{with probability } 1 - \frac{k}{N+1} \end{cases} .$$

Since X_k is a Bernoulli random variable (or by direct calculation), it follows that

$$E[X_k] = \frac{k}{N+1}, \qquad \text{var}[X_k] = \frac{k}{N+1}\frac{N+1-k}{N+1}.$$

The total number of white balls extracted is just

$$X = X_1 + \cdots + X_N.$$

This implies

$$E[X] = \sum_{k=1}^{N} E[X_k] = \sum_{k=1}^{N} \frac{k}{N+1} = \frac{N}{2}.$$

This can also be easily obtained from the complete symmetry of the problem between white and black balls.

For the variance, since all the X_i are independent, if follows that

$$\text{var}[X] = \sum_{k=1}^{N} \text{var}[X_k] = \sum_{k=1}^{N} \frac{k}{N+1}\frac{N+1-k}{N+1} = \frac{1}{6}\frac{N(N+2)}{N+1}.$$

(b) The probability of obtaining all white balls, except a black one from urn k is equal to

$$\frac{N+1-k}{N+1} \prod_{j \neq k} \frac{j}{N+1},$$

since the first term corresponds to the probability of "failure" in urn k, and "success" in all the other urns. This expression can be simplified (by multiplying and dividing by the missing term) to

$$\frac{N+1-k}{N+1}\frac{N+1}{k} \prod_{j=1}^{N} \frac{j}{N+1} = \frac{N+1-k}{k}\frac{N!}{(N+1)^N}.$$

(c) Define the events $B = \{\text{only one black ball was extracted}\}$ and $B_k = \{\text{the ball extracted from urn } k \text{ is black}\}$. Using the results of the previous exercise (and cancelling common terms), we have

$$P(B_k|B) = \frac{P(B \cap B_k)}{P(B)} = \frac{P(B \cap B_k)}{\sum_{j=1}^{N} P(B \cap B_j)} = \frac{\frac{N+1-k}{k}}{\sum_{j=1}^{N} \frac{N+1-j}{j}}.$$

P3 Here is a simple way of obtaining the answer. This morning, the backpack contained:

{Old Apple} with probability $\frac{1}{2}$
{Old Orange} with probability $\frac{1}{2}$

After adding an apple, your backpack contains

{Old Apple, New Apple} with probability $\frac{1}{2}$
{Old Orange, New Apple} with probability $\frac{1}{2}$

Now, you grab a fruit at random. The possible states, all equally likely, are then:

Hand	Backpack	Probability
Old Apple	New Apple	$\frac{1}{4}$
New Apple	Old Apple	$\frac{1}{4}$
Old Orange	New Apple	$\frac{1}{4}$
New Apple	Old Orange	$\frac{1}{4}$

Since the extracted fruit was an apple, the third case can be discarded. Of the remaining cases, in two of them the fruit in the backpack is an apple. Therefore, the desired probability is

$$\frac{P(\text{favorable cases})}{P(\text{possible cases})} = \frac{\frac{1}{4}+\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \frac{2}{3}.$$