Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2005)

Problem Set 6 Solutions
Due: March 30, 2005

1. Let $Y = X^2 + 2X + 1$. To find the PDF of Y, we first have to find its CDF. If $y \geq 0$,

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) \\ &= \mathbf{P}(X^2 + 2X + 1 \leq y) \\ &= \mathbf{P}((X+1)^2 \leq y) \\ &= \mathbf{P}(-\sqrt{y} \leq (X+1) \leq \sqrt{y}) \\ &= F_X(\sqrt{y} - 1) - F_X(-\sqrt{y} - 1), \end{aligned}$$

and, otherwise, it is 0. So, we have

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy}\Big(F_X(\sqrt{y} - 1) - F_X(-\sqrt{(y)} - 1)\Big) \\ &= \frac{d}{d(\sqrt{y}-1)} F_X(\sqrt{y} - 1)\frac{d}{dy}(\sqrt{y} - 1) - \frac{d}{d(-\sqrt{y}-1)} F_X(-\sqrt{y} - 1)\frac{d}{dy}(-\sqrt{y} - 1) \\ &= \frac{1}{2\sqrt{y}} f_X(\sqrt{y} - 1) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y} - 1). \end{aligned}$$

Note that the above only holds if $y \geq 0$, which makes sense since Y is a perfect square and is never negative. If $y < 0$, then $f_Y(y) = 0$.

To specialize our answer to the case where X is a uniformly distributed between 0 and 1, note that $f_X(-\sqrt{y} - 1) = 0$ for all values of y. Also, $f_X(\sqrt{y} - 1) = 1$ for $0 \leq \sqrt{y} - 1 \leq 1$ and 0 otherwise. Therefore, we have

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y} - 1),$$

and

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & \text{if } 1 \leq y \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

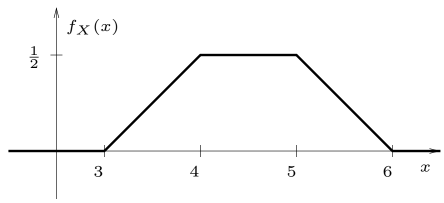
2. If $3 \leq z \leq 6$, we have

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx \\ &= \int_{\max(0,z-4)}^{\min(2,z-3)} \frac{1}{2}\, dx \\ &= (\min(2, z-3) - \max(0, z-4))/2. \end{aligned}$$

The PDF of $X + Y$ is then

$$f_{X+Y}(z) = \begin{cases} (z-3)/2, & \text{if } 3 \leq z < 4, \\ 1/2, & \text{if } 4 \leq z < 5, \\ (6-z)/2, & \text{if } 5 \leq z \leq 6, \\ 0, & \text{otherwise.} \end{cases}$$

The sketch of this PDF follows.



3. Let X and Y be the number of flips until Alice and Bob stop, respectively. Thus, $X+Y$ is the total number of flips until both stop. The random variables X and Y are independent geometric random variables with parameters 1/4 and 3/4, respectively. By convolution, we have

$$\begin{aligned} p_{X+Y}(j) &= \sum_{k=-\infty}^{\infty} p_X(k)p_Y(j-k) \\ &= \sum_{k=1}^{j-1}(1/4)(3/4)^{k-1}(3/4)(1/4)^{j-k-1} \\ &= \frac{1}{4^j}\sum_{k=1}^{j-1}3^k \\ &= \frac{1}{4^j}\left(\frac{3^j-1}{3-1}-1\right) \\ &= \frac{3}{2}\frac{(3^{j-1}-1)}{4^j}, \end{aligned}$$

if $j \geq 2$, and 0 otherwise. (Even though $X+Y$ is *not* geometric, it roughly behaves like one with parameter 3/4.)

4. Let Y and Z be independent exponential random variables with parameters 1 and 3, respectively. Then, we have

$$M_X(s) = M_Y(s)M_Z(s).$$

Hence, $X = Y + Z$. By convolution, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_Z(z)f_Y(x-z)\,dz \\ &= \int_0^x 3e^{-3z}e^{-(x-z)}\,dz \\ &= 3e^{-x}\int_0^x e^{-2z}\,dz \\ &= \frac{3}{2}(e^{-x}-e^{-3x}). \end{aligned}$$

if $x \geq 0$, and 0 otherwise. (Even though X is *not* exponential, it roughly behaves like one with parameter 1.)

5. (a) By linearity of expectation and since X_i have the same distribution as X, we have

$$\mathbf{E}[W] = 4\mathbf{E}[X].$$

Since, in addition, $X_1, X_2, \ldots$ are independent, by the linearity of the sum of independent random variables, we have

$$\text{var}(W) = 4\text{var}(X).$$

(b) Since $V = 0.25W$, by linearity of expectation, the variance of a linear function of a random variable, and the previous result, we have

$$\mathbf{E}[V] = \mathbf{E}[X], \text{var}(V) = \text{var}(X)/4.$$

(c) We have $U = W + Y$. Assuming the random variables $Y, X_1, X_2, \ldots$ are independent, W is independent of Y, since functions of different independent random variables are independent. Therefore, by linearity of expectation, the variance of the sum of independent random variables, and a previous result, we have

$$\mathbf{E}[U] = 4\mathbf{E}[X] + \mathbf{E}[Y], \text{var}(U) = 4\text{var}(X) + \text{var}(Y).$$

(d) By linearity of expectation, we have

$$\mathbf{E}[R] = 4\mathbf{E}[X] - \mathbf{E}[Y].$$

Since, in addition, X and Y are independent, by the variance of a sum of independent random variables and the variance of a linear function of a random variable, we obtain

$$\text{var}(R) = 16\text{var}(X) + \text{var}(Y).$$

(e) The transform of Q corresponds to that of a sum of 5 iid random variables with the same PMF as X. Hence, by the uniqueness property of transforms, linearity of expectation and the variance of a sum of independent random variables, we have

$$\mathbf{E}[Q] = 5\mathbf{E}[X], \text{var}(Q) = 5\text{var}(X).$$

(f) The transform of H corresponds to that of a sum of 5 independent random variables, 2 of them having the same PMF as X and 3 of them the same PMF as Y. By the uniqueness property of transform, linearity of expectation and the variance of a sum of independent random variables, we get

$$\mathbf{E}[H] = 2\mathbf{E}[X] + 3\mathbf{E}[Y], \text{var}(H) = 2\text{var}(X) + 3\text{var}(Y).$$

(g) By the transform of G and the uniqueness property of transforms, we have $G = X + 6$. By linearity of expectation and the variance of a linear function of a random variable, we obtain

$$\mathbf{E}[G] = \mathbf{E}[X] + 6, \text{var}(G) = \text{var}(X).$$

(h) By the transform of D and the uniqueness property of transforms, we have $D = 7X$. Therefore, by the linearity of expectation and the variance of a linear function of a random variable, we obtain

$$\mathbf{E}[D] = 7\mathbf{E}[X], \text{var}(D) = 49\text{var}(X).$$

6. (a) By comparing $M_X(s)$ with the definition of the transform, we see that X has the following PMF:

$$p_X(s) = \begin{cases} \frac{1}{8}, & \text{if } x = 1, \\ \frac{1}{2}, & \text{if } x = 2, \\ \frac{5}{24}, & \text{if } x = 4, \\ \frac{1}{6}, & \text{if } x = 6, \\ 0, & \text{otherwise.} \end{cases}$$

(b) From the total expectation theorem and the transform of the Poisson random variable,

$$\begin{aligned} \mathbf{E}[e^{sY}] &= \sum_{k \in \{1,2,4,6\}} p_X(k)\mathbf{E}[e^{sY}|X = k] \\ &= \sum_{k \in \{1,2,4,6\}} p_X(k)e^{k(e^s-1)} \\ &= \frac{1}{8}e^{(e^s-1)} + \frac{1}{2}e^{2(e^s-1)} + \frac{5}{24}e^{4(e^s-1)} + \frac{1}{6}e^{6(e^s-1)}. \end{aligned}$$

Note that we could have also used the law of iterated expectations, which we will see later.

(c)

$$\begin{aligned} \mathbf{E}[Y] &= \frac{d}{ds}M_Y(y)\bigg|_{s=0} \\ &= (1)\frac{1}{8}e^{(e^s-1)+s} + (2)\frac{1}{2}e^{2(e^s-1)+s} + (4)\frac{5}{24}e^{4(e^s-1)+s} + (6)\frac{1}{6}e^{6(e^s-1)+s}\bigg|_{s=0} \\ &= \left(\frac{1}{8}\right)(1) + \left(\frac{1}{2}\right)(2) + \left(\frac{5}{24}\right)(4) + \left(\frac{1}{6}\right)(6) \\ &= 2 + \frac{23}{24}. \end{aligned}$$

The above answer can also be found by simply noting that $\mathbf{E}[Y|X] = X$ from the property of the Poisson random variable.
We can find the second moment of Y by differentiating the transform twice and then obtain the variance from $\mathbf{E}[Y^2] - (\mathbf{E}[Y])^2$.

$$\begin{aligned} \mathbf{E}[Y^2] &= \frac{d^2}{ds^2}M_Y(y)\bigg|_{s=0} \\ &= \left(\frac{1}{8}\right)(1)(2) + \left(\frac{1}{2}\right)(2)(3) + \left(\frac{5}{24}\right)(4)(5) + \left(\frac{1}{6}\right)(6)(7) \\ &= 14 + \frac{5}{12}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{var}[Y] &= \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 \\ &= 14 + \frac{5}{12} - \left(2 + \frac{23}{24}\right)^2 \\ &= 5 + \frac{383}{576}. \end{aligned}$$

An alternative method is to use the law of total variance, which we will see later (see page 229 of the textbook).

7. The transforms associated with X, Y and Z are as follows:

$$\begin{aligned} M_X(s) &= e^{\frac{\sigma_X^2 s^2}{2}}, \\ M_Y(s) &= e^{\frac{\sigma_Y^2 s^2}{2}}, \text{ and} \\ M_Z(s) &= e^{\frac{\sigma_Z^2 s^2}{2}}. \end{aligned}$$

We know that $\text{var}(aX) = a^2\sigma_X^2$, so what aX is a zero-mean Gaussian random variable with the above variance. Moreover, since X, Y and Z are independent random variables, we have

$$\begin{aligned} M_Q(s) &= \mathbf{E}[e^{sQ}] \\ &= \mathbf{E}[e^{s(aX+bY+cZ)}] \\ &= \mathbf{E}[e^{asX}]\mathbf{E}[e^{bsY}]\mathbf{E}[e^{csZ}] \quad \text{due to independence} \\ &= e^{\frac{\sigma_X^2(as)^2}{2}} e^{\frac{\sigma_Y^2(bs)^2}{2}} e^{\frac{\sigma_Z^2(cs)^2}{2}} \\ &= e^{\frac{(a^2\sigma_X^2+b^2\sigma_Y^2+c^2\sigma_Z^2)s^2}{2}} \end{aligned}$$

Only Gaussian random varibles are associated with a transform of the form of $M_Q(s)$. Therefore, we conclude that the distribution of Q must be $N(0, a^2\sigma_X^2 + b^2\sigma_Y^2 + c^2\sigma_Z^2)$, and its PDF is given by

$$f_Q(q) = \frac{1}{\sqrt{2\pi(a^2\sigma_X^2 + b^2\sigma_Y^2 + c^2\sigma_Z^2)}} e^{-\frac{q^2}{2(a^2\sigma_X^2+b^2\sigma_Y^2+c^2\sigma_Z^2)}}.$$

Without using transforms, we would have to find the PDF of Q by doing convolution twice. Let us first define the random variable $W = aX + bY$. First note that the CDF of W is

$$\begin{aligned} F_W(w) &= \int_{-\infty}^{\infty}\int_{-\infty}^{(w-ax)/b} f_X(x)f_Y(y)\,dy\,dx \\ &= \int_{-\infty}^{\infty} f_X(x)F_Y((w-ax)/b)\,dx. \end{aligned}$$

Taking derivatives, we then get that the PDF of W is given by

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x)f_Y((w-ax)/b)\frac{1}{b}\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_X}e^{-x^2/(2\sigma_X^2)}\frac{1}{\sqrt{2\pi}\sigma_Y}e^{-((w-ax)/b)^2/(2\sigma_Y^2)}\frac{1}{b}\,dx \\ &= \frac{1}{2\pi b\sigma_X\sigma_Y}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2/\sigma_X^2+((w-ax)/b)^2/\sigma_Y^2)}\,dx. \end{aligned}$$

The second factor in the exponential can be expressed, by completing the square, as

$$\begin{aligned}
\frac{x^2}{\sigma_X^2} + \frac{((w-ax)/b)^2}{\sigma_Y^2} &= \frac{b^2\sigma_Y^2 x^2 + \sigma_X^2 w^2 - 2\sigma_X^2 wax + a^2\sigma_X^2 x^2}{b^2\sigma_X^2\sigma_Y^2} \\
&= \frac{(a^2\sigma_X^2 + b^2\sigma_Y^2)x^2 - 2xa\sigma_X^2 w + \sigma_X^2 w^2}{b^2\sigma_X^2\sigma_Y^2} \\
&= \left(\frac{a^2\sigma_X^2 + b^2\sigma_Y^2}{b^2\sigma_X^2\sigma_Y^2}\right)\left(x^2 - 2x\frac{a\sigma_X^2 w}{a^2\sigma_X^2 + b^2\sigma_Y^2} + \frac{\sigma_X^2 w^2}{a^2\sigma_X^2 + b^2\sigma_Y^2}\right) \\
&= \left(\frac{a^2\sigma_X^2 + b^2\sigma_Y^2}{b^2\sigma_X^2\sigma_Y^2}\right)\left(x^2 - 2x\frac{a\sigma_X^2 w}{a^2\sigma_X^2 + b^2\sigma_Y^2} + \left(\frac{a\sigma_X^2 w}{a^2\sigma_X^2 + b^2\sigma_Y^2}\right)^2 + \right. \\
&\qquad \left. - \left(\frac{a\sigma_X^2 w}{a^2\sigma_X^2 + b^2\sigma_Y^2}\right)^2 + \frac{\sigma_X^2 w^2}{a^2\sigma_X^2 + b^2\sigma_Y^2}\right) \\
&= \left(\frac{a^2\sigma_X^2 + b^2\sigma_Y^2}{b^2\sigma_X^2\sigma_Y^2}\right)\left(\left(x - \frac{a\sigma_X^2 w}{a^2\sigma_X^2 + b^2\sigma_Y^2}\right)^2 + \frac{b^2\sigma_X^2\sigma_Y^2 w^2}{(a^2\sigma_X^2 + b^2\sigma_Y^2)^2}\right) \\
&= \frac{\left(x - \frac{a\sigma_X^2 w}{a^2\sigma_X^2 + b^2\sigma_Y^2}\right)^2}{\left(\frac{b^2\sigma_X^2\sigma_Y^2}{a^2\sigma_X^2 + b^2\sigma_Y^2}\right)} + \frac{w^2}{a^2\sigma_X^2 + b^2\sigma_Y^2}.
\end{aligned}$$

To simplify notation, let

$$\mu = \frac{a\sigma_X^2 w}{a^2\sigma_X^2 + b^2\sigma_Y^2},$$

$$\sigma^2 = \frac{b^2\sigma_X^2\sigma_Y^2}{a^2\sigma_X^2 + b^2\sigma_Y^2},$$

and

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2.$$

Continuing our derivation of the PDF of W above, we then have

$$\begin{aligned}
f_W(w) &= \frac{1}{2\pi b\sigma_X\sigma_Y}\int_{-\infty}^{\infty} e^{-\frac{1}{2}((x-\mu)^2/\sigma^2 + w^2/\sigma_W^2)}\,dx \\
&= \frac{1}{2\pi b\sigma_X\sigma_Y} e^{-w^2/(2\sigma_W^2)} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\,dx \\
&= \frac{1}{2\pi b\sigma_X\sigma_Y} e^{-w^2/(2\sigma_W^2)}\sqrt{2\pi}\sigma \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}\,dx \\
&= \frac{1}{2\pi b\sigma_X\sigma_Y} e^{-w^2/(2\sigma_W^2)}\sqrt{2\pi}\sigma \\
&= \frac{1}{\sqrt{2\pi(a^2\sigma_X^2 + b^2\sigma_Y^2)}} e^{-\frac{1}{2}\frac{w^2}{(a^2\sigma_X^2 + b^2\sigma_Y^2)}},
\end{aligned}$$

where the second-to-last equality results because the integral evaluates to 1, since we are integrating the PDF of a normal random variable with mean μ and variance σ^2 over all its possible values. The last equality results after substituting the expression for σ and simplifying.

From our derivation of the PDF of W, we conclude that if $W = aX + bY$, where $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$, then $W \sim N(0, a^2\sigma_X^2 + b^2\sigma_Y^2)$. Therefore, if we apply this result again to the random variable $Q = aX + bY + cZ = W + cZ$, we get that $Q \sim N(0, 1^2(a^2\sigma_X^2 + b^2\sigma_Y^2) + c^2\sigma_Z^2)$, or, simplifying, $Q \sim N(0, a^2\sigma_X^2 + b^2\sigma_Y^2 + c^2\sigma_Z^2)$. Hence, the PDF of Q is

$$f_Q(q) = \frac{1}{\sqrt{2\pi(a^2\sigma_X^2 + b^2\sigma_Y^2 + c^2\sigma_Z^2)}} e^{-\frac{1}{2}\frac{q^2}{(a^2\sigma_X^2 + b^2\sigma_Y^2 + c^2\sigma_Z^2)}}.$$

8. Without using transforms, we know that the PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have

$$\begin{aligned} \mathbf{E}[X] &= \int_a^b \frac{x}{b-a}\,dx \\ &= \frac{x^2}{2(b-a)} \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{b+a}{2}. \end{aligned}$$

Using the transform, we know that

$$M_X(s) = \frac{1}{b-a} \cdot \frac{e^{sb} - e^{sa}}{s}$$

(see page 220 in the textbook). Now, evaluating its derivative at 0, we get

$$\begin{aligned} \mathbf{E}[X] &= \frac{d}{ds} M_X(s)\Big|_{s=0} \\ &= \frac{1}{b-a} \cdot \frac{(sb-1)e^{sb} - (sa-1)e^{sa}}{s^2}\Bigg|_{s=0} \\ &= 0/0, \end{aligned}$$

which has an indeterminate form. Applying L'Hôpital's rule, we get

$$\begin{aligned}
\mathbf{E}[X] &= \frac{1}{b-a} \cdot \left. \frac{be^{sb} + (sb-1)be^{sb} - ae^{sa} - (sa-1)ae^{sa}}{2s} \right|_{s=0} \\
&= \frac{1}{b-a} \cdot \left. \frac{sb^2e^{sb} - sa^2e^{sa}}{2s} \right|_{s=0} \\
&= \frac{1}{b-a} \cdot \left. \frac{b^2e^{sb} - a^2e^{sa}}{2} \right|_{s=0} \\
&= \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} \\
&= \frac{1}{b-a} \cdot \frac{(b-a)(b+a)}{2} \\
&= \frac{(b+a)}{2}.
\end{aligned}$$