Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Tutorial 3 Answers
Week of February 21, 2005

1. We are given the following information:

$$p_K(k) = \begin{cases} 1/4, & \text{if } k = 1,2,3,4; \\ 0, & \text{otherwise} \end{cases}$$

$$p_{N|K}(n \mid k) = \begin{cases} 1/k, & \text{if } n = 1,\ldots,k; \\ 0, & \text{otherwise} \end{cases}$$

(a) We use the fact that $p_{N,K}(n,k) = p_{N|K}(n \mid k)p_K(k)$ to arrive at the following joint PMF:

$$p_{N,K}(n,k) = \begin{cases} 1/(4k), & \text{if } k = 1,2,3,4 \text{ and } n = 1,\ldots,k; \\ 0, & \text{otherwise} \end{cases}$$

(b) The marginal PMF $p_N(n)$ is given by the following formula:

$$p_N(n) = \sum_k p_{N,K}(n,k) = \sum_{k=n}^{4} \frac{1}{4k}$$

On simplification this yields

$$p_N(n) = \begin{cases} 1/4 + 1/8 + 1/12 + 1/16 = 25/48, & n = 1; \\ 1/8 + 1/12 + 1/16 = 13/48, & n = 2; \\ 1/12 + 1/16 = 7/48, & n = 3; \\ 1/16 = 3/48, & n = 4; \\ 0, & \text{otherwise.} \end{cases}$$

(c) The conditional PMF is

$$p_{K|N}(k \mid 2) = \frac{p_{N,K}(2,k)}{p_N(2)} = \begin{cases} 6/13, & k = 2; \\ 4/13, & k = 3; \\ 3/13, & k = 4; \\ 0, & \text{otherwise.} \end{cases}$$

(d) Let A be the event $2 \le N \le 3$. We first find the conditional PMF of K given A.

$$\begin{aligned} p_{K|A}(k) &= \frac{\mathbf{P}(K = k, A)}{\mathbf{P}(A)} \\ \mathbf{P}(A) &= p_N(2) + p_N(3) = \frac{5}{12} \\ \mathbf{P}(K = k, A) &= \begin{cases} \frac{1}{8}, & k = 2; \\ \frac{1}{12} + \frac{1}{12}, & k = 3; \\ \frac{1}{16} + \frac{1}{16}, & k = 4; \\ 0, & \text{otherwise} \end{cases} \\ p_{K|A}(k) &= \begin{cases} \frac{3}{10}, & k = 2; \\ \frac{2}{5}, & k = 3; \\ \frac{3}{10}, & k = 4; \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Because the conditional PMF of K given A is symmetric around $k = 3$, we know $\boxed{\mathbf{E}[K \mid A] = 3}$. We now find the conditional variance of K given A.

$$\begin{aligned} var(K \mid A) &= \mathbf{E}[(K - \mathbf{E}[K \mid A])^2 \mid A] \\ &= \frac{3}{10} \cdot (2-3)^2 + \frac{2}{5} \cdot 0 + \frac{3}{10} \cdot (4-3)^2 \\ &= \boxed{\tfrac{3}{5}} \end{aligned}$$

2. (a) An easy way to derive $p_{X,Y,Z}(x,y,z)$ is in sequential terms as $p_X(x) \cdot p_{Y,Z|X}(y,z|x)$. Note $p_X(x)$ is geometric with parameter p. Conditioned on X even, $(Y,Z) = (0,0)$ with probability 1. Conditioned on X odd, $p_{Y,Z|X}(y,z) = \frac{1}{4}$ for $(y,z) \in \{(0,0),(0,2),(2,0),(2,2)\}$.

$$p_{X,Y,Z}(x,y,z) = \begin{cases} \frac{1}{4}p(1-p)^{x-1}, & \text{if } x \text{ is odd and } (y,z) \in \{(0,0),(0,2),(2,0),(2,2)\} \\ p(1-p)^{x-1}, & \text{if } x \text{ is even and } (y,z) = (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

(b) (i) No. Notice that even though conditional on X (i.e. given a realization, x, of random variable X), the random variables Y and Z are independent (that's why they look "regular"), Y and Z are not independent. Given Y, the distribution over Z changes (i.e. if Y is 2, Z is equally likely to be 0 or 2; however if Y is 0, Z is more likely to be 0).

(ii) Yes. Given $Z = 2$, if we are further given $X = x$, Y is equally likely to take on the value 0 or 2.

(iii) No. Given $Z = 0$, if we are further given $X = x$, then if x is even, Y must be 0, whereas if x is odd, Y is equally likely to take on 0 or 2.

(iv) Yes. Given $Z = 2$, if we are further given $X = x$, $Z = 2$ still holds (i.e. with probability 1)! Double conditioning has no effect.

(c) If $X = 5$, then Y and Z are uniformly distributed on the set S specified in the problem statement, so $Y + Z$ takes the values 0 and 4 with probability $\frac{1}{4}$, and takes the value 2 with probability $\frac{1}{2}$. This PMF is symmetric about 2, so the mean value of $Y + Z$ is evidently 2. Hence the variance is

$$(0-2)^2\frac{1}{4} + (4-2)^2\frac{1}{4} = 2.$$