Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2005)

Problem Set 2
Due: February 16, 2005

1. Imagine a drunk tightrope walker, who manages to keep his balance, but takes a step forward with probability p and takes a step back with probability $(1-p)$.

   a) What is the probability that after 2 steps the tightrope walker will be at the same place on the rope?

   b) What is the probability that after three steps, the tightrope walker will be one step forward from where he began?

   c) Given that after three steps he has managed to move ahead one step, what is the probability that the first step he took was a step forward?

2. A ball is in any one of n boxes. It is in the ith box with probability P_i. If the ball is in box i, a search of that box will uncover it with probability α_i. Show that the conditional probability that the ball is in box j, given that a search in box i did *not* uncover it, is

$$\frac{P_j}{1-\alpha_i P_i} \quad \text{if } j \neq i \qquad \text{and} \qquad \frac{(1-\alpha_i)P_i}{1-\alpha_i P_i} \quad \text{if } j = i.$$

3. Fischer and Spassky play a sudden death chess match. Each game ends up with either a win by Fischer, this happens with probability p, a win for Spassky, this happens with probability q, or a draw, this happens with probability $1-p-q$. The match continues until one of the players wins a game (and the match).

   (a) What is the probability that Fischer will win the last game of the match?
   (b) Given that the match lasted no more than 5 games, what is the probability that Fischer won in the first game?
   (c) Given that the match lasted no more than 5 games, what is the probability that Fischer won the match?
   (d) Given that Fischer won the match, what is the probability that he won at or before the 5th game?

4. Consider some sample space Ω. Suppose $A, B \subset \Omega$. Prove or disprove the following:

   (a) A, B independent implies A, B^c independent.
   (b) A, B independent implies A^c, B independent.
   (c) A, B independent implies A^c, B^c independent.

5. You are lost in the campus of MIT, where the population is entirely composed of brilliant students and absent-minded professors. The students comprise two-thirds of the population, and any one student gives a correct answer to a request for directions with probability $\frac{3}{4}$. (Assume answers to repeated questions are independent, even if the question and the person asked are the same.) If you ask a professor for directions, the answer is always false.

(a) You ask a passer-by whether the exit from campus is East or West. The answer is East. What is the probability this is correct?

(b) You ask the same person again, and receive the same reply. Show that the probability that this second reply is correct is $\frac{1}{2}$.

(c) You ask the same person again, and receive the same reply. What is the probability that this third reply is correct?

(d) You ask for the fourth time, and receive the answer East again. Show that the probability it is correct is $\frac{27}{70}$.

(e) Show that, had the fourth answer been West instead, the probability that East is nevertheless correct is $\frac{9}{10}$.

Your friend, Ima Nerd, happens to be in the same position as you are, only she has reason to believe a-priori that, with probability ϵ, East is the correct answer.

(f) Show that whatever answer is first received, Ima continues to believe that East is correct with probability ϵ.

(g) Show that if the first two replies are the same (that is, either WW or EE), Ima continues to believe that East is correct with probability ϵ.

(h) Show that after three like answers, Ima will calculate as follows (in the obvious notation):

$$\mathbf{P}(\text{East correct}|EEE) = \frac{9\epsilon}{11 - 2\epsilon}, \qquad \mathbf{P}(\text{East correct}|WWW) = \frac{11\epsilon}{9 + 2\epsilon}.$$

Evaluate these when $\epsilon = \frac{9}{20}$.