

## LECTURE 6

- **Readings:** Sections 2.4-2.6

### Lecture outline

- Review PMF, expectation, variance
- Conditional PMF
- Geometric PMF
- Total expectation theorem
- Joint PMF of two random variables

## Review

- Random variable  $X$ : function from sample space to the real numbers
- PMF (for discrete random variables):  
 $p_X(x) = \mathbf{P}(X = x)$
- Expectation:

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

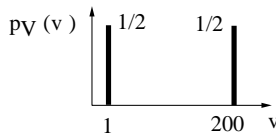
$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

- $\mathbf{E}[X - \mathbf{E}[X]] =$

$$\begin{aligned} \text{var}(X) &= \mathbf{E}[(X - \mathbf{E}[X])^2] \\ &= \sum_x (x - \mathbf{E}[X])^2 p_X(x) \\ &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \end{aligned}$$

### Average speed vs. average time

- Traverse a 200 mile distance at constant but random speed  $V$

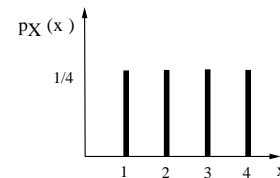


- $d = 200$ ,  $T = t(V) = 200/V$   
 $\mathbf{E}[V] = 1 \cdot (1/2) + 200 \cdot (1/2) = 100.5$   
 $\mathbf{E}[T] = \mathbf{E}[t(V)] = \sum_v t(v) p_V(v)$   
 $= \frac{200}{1} \cdot \frac{1}{2} + \frac{200}{200} \cdot \frac{1}{2} = 100.5$
- $\mathbf{E}[T] \cdot \mathbf{E}[V] \neq 200 = d = \mathbf{E}[TV]$
- $\mathbf{E}[T] \neq 200/\mathbf{E}[V]$ .  
 $\text{var}(V) = \sum_v (v - \mathbf{E}[V])^2 p_V(v)$   
 $= (1 - 100.5)^2 \frac{1}{2} + (200 - 100.5)^2 \frac{1}{2} \approx 10,000$
- standard deviation  $\sigma_V = \sqrt{\text{var}(V)} \approx 100$

### Conditional expectation

- $p_{X|A}(x) = \mathbf{P}(X = x | A)$

$$\mathbf{E}[X | A] = \sum_x x p_{X|A}(x)$$



- $\mathbf{E}[X | X \geq 2] =$

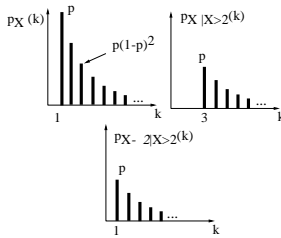
## Geometric PMF

- $X$ : number of independent coin tosses until first head

$$p_X(k) = (1-p)^{k-1}p, \quad k = 1, 2, \dots$$

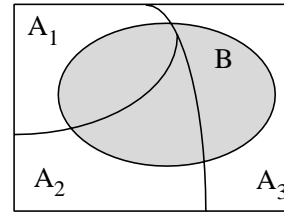
$$E[X] = \sum_{k=1}^{\infty} kp_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

- Memoryless property: Given that  $X > 2$ , the r.v.  $X - 2$  has same geometric PMF



## Total Expectation theorem:

- Partition of sample space into disjoint events  $A_1, A_2, \dots, A_n$



$$P(B) = P(A_1)P(B | A_1) + \dots + P(A_n)P(B | A_n)$$

$$E[X] = P(A_1)E[X | A_1] + \dots + P(A_n)E[X | A_n]$$

- Geometric example:

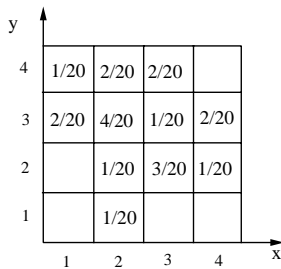
$$A_1 : \{X = 1\}, \quad A_2 : \{X > 1\}$$

$$E[X] = P(X = 1)E[X | X = 1] + P(X > 1)E[X | X > 1]$$

- Solve to get  $E[X] = 1/p$

## Joint PMFs

- $p_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$



- $\sum_x \sum_y p_{X,Y}(x,y) =$

- $p_X(x) = \sum_y p_{X,Y}(x,y)$

- $p_{X|Y}(x | y) = P(X = x | Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

- $\sum_x p_{X|Y}(x | y) =$