

## LECTURE 5

- **Readings:** Sections 2.1-2.3, start 2.4

### Lecture outline

- Random variables
- Probability mass function (pmf)
  - Service facility example
- Expectation

## Random variables

- An assignment of a value (number) to every possible outcome
- Mathematically: A function from the sample space to the real numbers
  - discrete or continuous
- Can have several random variables defined on the same sample space
- Notation:
  - random variable  $X$
  - numerical value  $x$

### Probability mass function (pmf)

- (“probability law”, “probability distribution”)
- Notation:

$$p_X(x) = \mathbf{P}(X = x)$$

- **Example:**  $X$ =number of coin tosses until first head

- assume independent tosses,  
 $\mathbf{P}(H) = p > 0$

$$\begin{aligned} p_X(k) &= \mathbf{P}(X = k) \\ &= \mathbf{P}(TT \cdots TH) \\ &= (1 - p)^{k-1}p, \quad k = 1, 2, \dots \end{aligned}$$

### How to compute a pmf $p_X(x)$

- collect all possible outcomes for which  $X$  is equal to  $x$
- add their probabilities
- repeat for all  $x$

- **Example:** Two independent throws of a fair tetrahedral die

$F$ : outcome of first throw

$S$ : outcome of second throw

$$L = \min(F, S)$$

	1	2	3	4
4				
3				
2				
1				
	1	2	3	4

$$p_L(2) =$$

## Binomial pmf

- $X$ : number of heads in  $n$  independent coin tosses
- $P(H) = p$
- Let  $n = 4$

$$\begin{aligned} p_X(2) &= P(HHTT) + P(HTHT) + P(HTTH) \\ &\quad + P(THHT) + P(THTH) + P(TTTH) \\ &= 6p^2(1-p)^2 \\ &= \binom{4}{2}p^2(1-p)^2 \end{aligned}$$

In general:

$$p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0, 1, \dots, n$$

## Service facility design

- $n$ : customers
- $p$ : prob. customer requires service
- $s$ : no. of service persons
- $X$ : no. of service requests (r.v.)

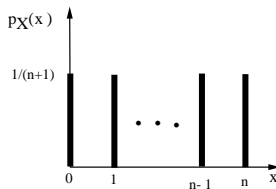
$$\begin{aligned} P(X > s) &= P(X = s + 1) + \dots + P(X = n) \\ &= p_X(s + 1) + \dots + p_X(n) \\ &= \sum_{i=s+1}^n \binom{n}{i}p^i(1-p)^{n-i} \end{aligned}$$

## Expectation

- Definition:

$$E[X] = \sum_x xp_X(x)$$

- Interpretations:
  - Center of gravity of pmf
  - Average in large number of repetitions of the experiment (to be substantiated later in this course)
- Example: Uniform on  $0, 1, \dots, n$



$$E[X] = 0 \times \frac{1}{n+1} + 1 \times \frac{1}{n+1} + \dots + n \times \frac{1}{n+1} =$$

## Properties of expectations

- Let  $X$  be a r.v. and let  $Y = g(X)$ 
  - Hard:  $E[Y] = \sum_y yp_Y(y)$
  - Easy:  $E[Y] = \sum_x g(x)p_X(x)$
- “Second moment”:  $E[X^2]$
- Caution: In general,  $E[g(X)] \neq g(E[X])$
- Variance:
 
$$\text{var}(X) = E[(X - E[X])^2] = \sum_x (x - E[X])^2 p_X(x)$$
- If  $\alpha$  is a constant:  $E[\alpha] =$
- $E[\alpha X] =$
- $E[\alpha X + \beta] =$