Recitation 10 Solutions
March 15, 2005

1. We need to show $p_Z(z) = p_Z(-z)$ where $Z = X + Y$ and X, Y are iid.

$$\begin{aligned} p_Z(z) &= \sum_x p_X(x) p_Y(z-x) \\ &= \sum_x p_X(x) p_X(z-x). \end{aligned}$$

Let $w = -x$.

$$\begin{aligned} p_Z(z) &= \sum_w p_X(-w) p_X(z+w) \\ &= \sum_w p_X(w) p_X(-(z+w)) \\ &= \sum_w p_X(w) p_X(-z-w)) \\ &= p_Z(-z) \end{aligned}$$

To show maximum value of $p_Z(z)$ is at $z = 0$,

$$\begin{aligned} p_Z(z) &= \sum_x p_X(x) p_X(z-x) \\ &\leq \left[\sum_x p_X(x)^2\right]^{\frac{1}{2}} \left[\sum_x (p_X(z-x))^2\right]^{\frac{1}{2}} \\ &= \sum_x p_X(x)^2 \\ &= \sum_x p_X(x) p_X(-x) \\ &= \sum_x p_X(x) p_X(0-x) \\ &= p_Z(0) \end{aligned}$$

Note: $\sum_x (p_X(z-x))^2 = \sum_x p_X(x)^2$. This is because in both cases, we are summing the squared values of the same PMF p_X.

2. Let X be the number of papers you hand in after receiving the first grade from your first paper until you receive a grade you had not seen. Let Y be the number of papers you hand in after receiving two different grades for the first time until you receive the remaining grade that you had not seen. X and Y are independent geometric random variables with parameters 2/3 and 1/3, respectively.

Then $Z = 1 + X + Y$ is the total number of papers you hand in until you received every grade at least once. By convolution, we have that the PMF of Z is, for $j \geq 3$,

$$\begin{aligned} p_Z(j) &= \sum_{k=-\infty}^{\infty} p_X(k)p_Y(j-1-k) \\ &= \sum_{k=1}^{j-2} (2/3)(1/3)^{k-1}(1/3)(2/3)^{j-1-k-1} \\ &= (2/3)^{j-1} \sum_{k=1}^{j-2} (1/2)^k \\ &= (2/3)^{j-1} \left(\frac{1-(1/2)^{j-1}}{1-1/2} - 1 \right) \\ &= \frac{2}{3^{j-1}}(2^{j-2}-1), \end{aligned}$$

and it is 0 otherwise.

3. (a) Let $X = X_1 + X_2 + ... + X_n$ where X_i is the amount of money won in the ith toss. $X_i = 1$ with probability $\binom{3}{2}p^2(1-p) + \binom{3}{3}p^3$. Let $q = \mathbf{P}(X_i = 1)$. X is a binomial random variable with the following PMF:

$$p_X(x) = \begin{cases} \binom{n}{x} q^x (1-q)^{n-x} & \text{for } 0 \leq x \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$p_Y(y) = \begin{cases} \binom{n}{y} q^y (1-q)^{n-y} & \text{for } 0 \leq y \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since X and Y are independent, the PMF of Z can be found by convolving the PMFs of X and Y:

$$\begin{aligned} p_Z(z) &= \sum_{k=-\infty}^{\infty} p_X(k)p_Y(z-k) \\ &= \sum_{k=\max(0,z-n)}^{\min(z,n)} \binom{n}{k} q^k (1-q)^{n-k} \binom{n}{z-k} q^{z-k}(1-q)^{n-z+k} \\ &= q^z (1-q)^{2n-z} \sum_{k=\max(0,z-n)}^{\min(z,n)} \binom{n}{k}\binom{n}{z-k} \end{aligned}$$

if $0 \leq z \leq 2n$, and $p_Z(z) = 0$ otherwise.

(c) Let A be the event Z is greater than or equal to \$10. Conditioned on A, X and Y are now dependent random variables and we cannot apply the convolution principle.

Instead,

$$\begin{aligned}
p_{Z|A}(z) &= \frac{\mathbf{P}(Z = z, Z \geq 10)}{\mathbf{P}(Z \geq 10)} \\
&= \frac{\mathbf{P}(Z = z)}{\mathbf{P}(Z \geq 10)} \\
&= \frac{p_Z(z)}{\sum_{z=10}^{2n} p_Z(z)}
\end{aligned}$$

for $z \geq 10$, and $p_{Z|A}(z) = 0$ otherwise.