Recitation 24 - Solutions
May 12, 2005

1. (a) $r_{12}(2) = p_{11}p_{12} + p_{12}p_{22} = .54$

   (b) To obtain the steady-state probabilities, solve the following linear system of equations: $\frac{3}{5}\pi_1 = \frac{3}{10}\pi_2$, $\pi_2 = \pi_3$, and $\pi_1 + \pi_2 + \pi_3 = 1$. We find that $\pi_1 = \frac{1}{5}$ and $\pi_2 = \pi_3 = \frac{2}{5}$.

   (c) In steady-state, $P(Y_n = 1) = \frac{3}{5}\pi_1 + \frac{1}{5}\pi_2 = \frac{1}{5}$.

   (d) No, Y_n is NOT a Markov Chain. Consider getting $Y_n = Y_{n+1} = 1$. If this is true, then $P(Y_{n+2} = 1) = 0$ since we cannot upward transition again. Thus, the chain will always need the information about the entire past, i.e. not Markov.

   (e) The reward, R, is uniform on $[0, 2]$ if the process is in state 1 and is uniform on $[0, 4]$ when the process is in state 2. Thus, $E[R] = p_{11}E[R|X_1 = 1]+p_{12}E[R|X_1 = 2] = \frac{2}{5}\cdot 1+\frac{3}{5}\cdot 2 = \frac{8}{5}$. In order to compute the variance, we need to first find the second moment of R. $E[R^2] = p_{11}E[R^2|X_1 = 1] + p_{12}E[R^2|X_1 = 2] = \frac{2}{5}(\frac{4}{12} + 1^2) + \frac{3}{5}(\frac{16}{12} + 2^2) = \frac{56}{15}$.
   Thus, $\text{var}(R) = \frac{56}{15} - (\frac{8}{5})^2 = \frac{88}{75}$.

   (f) No. As we showed in part (b), X_n takes the values 1, 2 and 3 with probabilities 1/5, 2/5 and 2/5 even as n goes to ∞, so it does not converge to a single number.

   (g) Yes, Z_n converges to 3 in probability, because the probability that there state 3 will be visited at least once increases to 1 as n increases. $\mathbf{P}(Z_n \neq 3) = \mathbf{P}(X_i \neq 3 \text{ for all } i \leq n) \to 0$ as $n \to \infty$.

   (h) Let X_i be Bernoulli with the probability of a success being .6. Let $A_k = \frac{X_1+...+X_k}{k}$ and hence $E[A_k] = E[X] = .6$ and $\text{var}(A_k) = \frac{\text{var}(X)}{k} = \frac{.24}{k}$. Thus, using the CLT,

$$P(.59 < A_k < .61) = P(\frac{.59 - .6}{\sqrt{\frac{.24}{k}}} < \frac{A_k - .6}{\sqrt{\frac{.24}{k}}} < P(\frac{.61 - .6}{\sqrt{\frac{.24}{k}}}) \approx 2\Phi(\frac{.01\sqrt{k}}{\sqrt{.24}}) - 1 \approx .96,$$

   by the problem statement. Thus, $\frac{.01\sqrt{k}}{\sqrt{.24}} \approx 2.06$ by the look-up table and hence $k \approx 10185$.

2. (a) Note that there is a possibility that Crocodile Dundee will arrive to the water hole at first and see a dingo. In the case, the amount of time he'll have to wait is 0. Otherwise, he'll have to wait until one arrives, and the amount of time he'll have to wait is exponential with rate 4. Let π_d denote the probability that when Crocodile Dundee arrives he sees a dingo at the water hole and π_w denote the probability he sees a wombat. Because the series of arrivals preceding the moment when Crocodile Dundee arrives also form a Poisson process (i.e. we can look at the arrivals in reverse and the interarrival times are still exponential), we have from part b) that $\pi_d = \frac{2}{3}$ and $\pi_w = \frac{1}{3}$.
   If we denote the amount of time until he sees a dingo as X, we have

$$\begin{aligned} E[X] &= E[X|\text{he arrives to hole occupied by a dingo}]P(\text{occupied by dingo}) + \\ &\quad E[X|\text{he arrives to hole occupied by a wombat}]P(\text{occupied by wombat}) \\ &= (0)\pi_d + E[exponential(4)]\pi_w \\ &= \frac{1}{12} \end{aligned}$$

(b) We again need to condition on who is occupying the hole when he arrives. Note that if Crocodile Dundee arrives at the water hole and a dingo is occupying the hole, the total amount of time that dingo spent at the hole is in fact Erlang order 2 by the random incidence phenomenon (there's an exponential amount of time left from when Crocodile Dundee arrives until another animal arrives, and there's an independent exponential amount of time from when the dingo first arrived to when Crocodile Dundee arrives). If Crocodile Dundee arrives when the hole is occupied by a wombat, then when the first dingo does come, the amount of time that dingo will spend will just be the interarrival time of a Poisson process of rate $2+4$, which is *exponential*(6). So if we call the amount of time spent by the first dingo seen by Crocodile Dundee Y, we have:

$$\begin{aligned} E[Y] &= E[Y|\text{he arrives to hole occupied by a dingo}]P(\text{occupied by dingo}) + \\ &\quad E[Y|\text{he arrives to hole occupied by a wombat}]P(\text{occupied by wombat}) \\ &= E[\text{Erlang-order2}(6)]\pi_d + E[\text{exponential}(6)]\pi_w \\ &= \frac{5}{18} \end{aligned}$$

(c) Let Y be the expected number of hours Crocodile Dundee spends at the water hole. Y is a sum of 900 independent interarrival times, each with an exponential distribution with parameter 6. The expected value of each interarrival times is $\frac{1}{6}$, and the variance of each is $\frac{1}{6^2} = \frac{1}{36}$. Therefore $E[Y] = 900(\frac{1}{6}) = 150$ and $\text{Var}(Y) = 900(\frac{1}{36}) = 25$.
$P(140 < Y < 160) = P(|Y - 150| < 10) = P(|Y - E[Y]| < 10)$.
By Chebyshev's inequality, $P(|Y - E[Y]| \geq 10) \leq \frac{\sigma^2}{10^2} = \frac{25}{100} = \frac{1}{4}$, so the probability that Crocodile stays at the water hole for between 140 and 160 hours is at least $1 - \frac{1}{4} = \frac{3}{4}$.

(d) Since Y is a sum of 900 independent, identically distributed random variables, the Central Limit Theorem implies that the cdf of $\frac{Y-150}{5}$ can be approximated by the cdf of $X \sim N(0,1)$, so

$$P(|Y - 150| < 10) = P\left(\frac{|Y-150|}{5} < 2\right) \approx P(-2 < X < 2) = 2(\Phi(2) - .5) = .954.$$

The probability that Crocodile stays at the water hole for between 140 and 160 hours is therefore approximately .954.