

## LECTURE 23

### Outline

- Approximating binomial distributions
- Strong law of large numbers

### CLT review

- $X_i$ : i.i.d., finite variance  $\sigma^2$
- $S_n = X_1 + \dots + X_n$
- $Z_n = \frac{(X_1 + \dots + X_n) - n\mathbf{E}[X]}{\sigma\sqrt{n}}$
- $Z$ : standard normal  
(zero mean, unit variance)
- **CLT:**  $\mathbf{P}(Z_n \leq c) \rightarrow \mathbf{P}(Z \leq c) = \Phi(c)$
- Approximation: treat  $S_n$  as if normal

### Example

- $n = 36$ ,  $p = 0.5$ ; find  $\mathbf{P}(S_n \leq 21)$

- Exact answer:

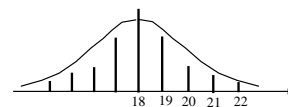
$$\sum_{k=0}^{21} \binom{36}{k} \left(\frac{1}{2}\right)^{36} = 0.8785$$

### Apply to binomial

- Fix  $p$ , where  $0 < p < 1$
- $X_i$ : Bernoulli( $p$ )
- $S_n = X_1 + \dots + X_n$ : Binomial( $n, p$ )
  - mean  $np$ , variance  $np(1-p)$
- $\frac{S_n - np}{\sqrt{np(1-p)}} \rightarrow$  standard normal CDF

### The 1/2 correction for binomial approximation

- $\mathbf{P}(S_n \leq 21) = \mathbf{P}(S_n < 22)$ ,  
because  $S_n$  is integer
- Compromise: consider  $\mathbf{P}(S_n \leq 21.5)$



## De Moivre–Laplace CLT (for binomial)

- When the 1/2 correction is used, CLT can also approximate the binomial p.m.f. (not just the binomial CDF)

$$P(S_n = 19) = P(18.5 \leq S_n \leq 19.5)$$

$$18.5 \leq S_n \leq 19.5 \iff$$

$$\frac{18.5 - 18}{3} \leq \frac{S_n - 18}{3} \leq \frac{19.5 - 18}{3} \iff$$

$$0.17 \leq Z_n \leq 0.5$$

$$P(S_n = 19) \approx P(0.17 \leq Z \leq 0.5)$$

$$= P(Z \leq 0.5) - P(Z \leq 0.17)$$

$$= 0.6915 - 0.5675$$

$$= 0.124$$

- Exact answer:

$$\binom{36}{19} \left(\frac{1}{2}\right)^{36} = 0.1251$$

## The strong law of large numbers

- Theorem: (SLLN)** If  $X_i$  are i.i.d., and  $E[|X|] < \infty$ , then

$$M_n = \frac{X_1 + \dots + X_n}{n} \rightarrow E[X]$$

with probability 1.

- One experiment:** Generate an infinite sequence  $X_1, X_2, \dots$  and the associated infinite sequence  $M_1, M_2, \dots$ 
  - Sample space: set of all infinite sequences
- Sequences that do not converge to  $E[X]$  are also possible, but collectively their probability is zero
- Example:  $X_i$  are Bernoulli( $p$ )
  - $M_n =$  fraction of successes in  $n$  trials
  - $M_n \rightarrow p$ , with probability 1

## Poisson vs. normal approximations of the binomial

- Poisson arrivals during unit interval equals: sum of  $n$  (independent) Poisson arrivals during  $n$  intervals of length  $1/n$ 
  - Let  $n \rightarrow \infty$ , apply CLT (??)
  - Poisson=normal (????)
- Binomial( $n, p$ )
  - $p$  fixed,  $n \rightarrow \infty$ : normal
  - $np$  fixed,  $n \rightarrow \infty, p \rightarrow 0$ : Poisson
- $p = 1/100, n = 100$ : Poisson
- $p = 1/10, n = 500$ : normal

## Convergence with probability one

- Let **one** experiment generate an **infinite** sequence of experimental values of  $Y_1, Y_2, \dots$ 
  - sample space: set of all sequences
- Def:**  $Y_n \rightarrow c$ , with probability 1, if the set of all sequences that converge to  $c$  has probability 1.
- It turns out that convergence w.p.1 always implies convergence in probability
- converse is not always true
  - Example: let  $Y_n = 1$  if customer arrives at time slot  $n$ 
    - Model of customer arrivals: during each interval  $2^k, \dots, 2^{k+1} - 1$ , exactly 1 customer arrives, each slot being equally likely
    - Check that  $Y_n \rightarrow 0$  in probability, but not with probability 1