LECTURE 12

- **Readings:** Section 4.1

Lecture outline

- Definition of transforms
- Why transforms?
- Moment generating properties
- Examples
- Transform of a sum of independent r.v.'s

Definition of transforms

$$M_X(s) = \mathbf{E}[e^{sX}]$$

- X discrete, pmf $p_X(x)$

$$M_X(s) = \mathbf{E}[e^{sX}] = \sum_x e^{sx} p_X(x)$$

- X continuous, pdf $f_X(x)$

$$M_X(s) = \mathbf{E}[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

- **Inversion theorem:**
  Know the transform
  $\Longrightarrow$ pmf or pdf uniquely determined

Why transforms

- A new kind of representation
- Sometimes convenient for:
  - calculations
  - analytical derivations
    and theorem proving

Moment generating properties

- Find moments without integrating

$$M_X(s) = \mathbf{E}[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

$$M_X(s)|_{s=0} = \mathbf{E}[e^{0X}] = 1$$

$$\left.\frac{d}{ds} M_X(s)\right|_{s=0} =$$

$$\left.\frac{d^n}{ds^n} M_X(s)\right|_{s=0} = \mathbf{E}[X^n]$$

Exponential pdf example

$(\lambda > 0)$

$$\begin{aligned} M_X(s) &= \lambda \int_0^\infty e^{sx} e^{-\lambda x}\, dx \\ &= \lambda \int_0^\infty e^{(s-\lambda)x}\, dx \\ &= \frac{\lambda}{\lambda - s} \end{aligned}$$

$$\mathbf{E}[X] = \frac{d}{ds} M_X(s)\Big|_{s=0} = \frac{\lambda}{(\lambda - s)^2}\Big|_{s=0} = \frac{1}{\lambda}$$

- Can also get $\mathbf{E}[X^2]$ etc. this way

Discrete random variables

- If X takes nonnegative integer values:

$$\begin{aligned} M_X(s) &= \mathbf{E}[e^{sX}] = \sum_x e^{sx} p_X(x) \\ &= p_X(0) + p_X(1)e^s + p_X(2)e^{2s} + \cdots \end{aligned}$$

- $M_X(s) = \dfrac{pe^s}{1 - (1-p)e^s}$
- Use $\dfrac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots$, for $|\alpha| < 1$
- $M_X(s) =$ $pe^s(1 + (1-p)e^s + (1-p)^2 e^{2s} + (1-p)^3 e^{3s} + \cdots)$
- $p_X(x) = p(1-p)^{x-1}, \qquad x = 1, 2, \ldots$ (geometric distribution)

Sums of independent random variables

- X, Y independent

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

- $W = X + Y$
- $M_W(s) = M_X(s) M_Y(s)$
- Add r.v.'s $\iff$ multiply transforms

Transform of normal

- General normal $N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- Transform: $\quad M_X(s) = e^{(s^2\sigma^2/2) + s\mu}$
- **Sum of independent normal:** $X \sim N(m_x, \sigma_x^2)$, $Y \sim N(m_y, \sigma_y^2)$, $W = X + Y$

$$\begin{aligned} M_W(s) &= M_X(s) M_Y(s) \\ &= e^{(s^2\sigma_x^2/2) + sm_x} e^{(s^2\sigma_y^2/2) + sm_y} \\ &= e^{(s^2(\sigma_x^2 + \sigma_y^2)/2) + s(m_x + m_y)} \end{aligned}$$