LECTURE 10

More on continuous r.v.s; derived distributions

- **Readings:** Section 3.6

Review

$$p_X(x) \qquad f_X(x)$$

$$p_{X,Y}(x,y) \qquad f_{X,Y}(x,y)$$

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \qquad f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$p_X(x) = \sum_y p_{X,Y}(x,y) \qquad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$F_X(x) = \mathbf{P}(X \le x)$$

$$\mathbf{E}[X], \quad \text{var}(X)$$

Buffon's needle

- Parallel lines at distance d
  Needle of length ℓ (assume $\ell < d$)
- Find $\mathbf{P}$(needle intersects one of the lines)
- $X \in [0, d/2]$: distance of needle midpoint to nearest line
- Model: X, Θ uniform, independent

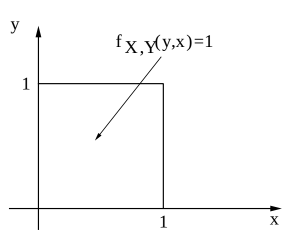
  $f_{X,\Theta}(x,\theta) = \qquad\qquad 0 \le x \le d/2,\ 0 \le \theta \le \pi/2$

- Intersect if $X \le \dfrac{\ell}{2} \sin \Theta$

$$\begin{aligned}
\mathbf{P}\left(X \le \frac{\ell}{2}\sin\Theta\right) &= \int\int_{x \le \frac{\ell}{2}\sin\theta} f_X(x) f_\Theta(\theta)\, dx\, d\theta \\
&= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{(\ell/2)\sin\theta} dx\, d\theta \\
&= \frac{4}{\pi d}\int_0^{\pi/2}\frac{\ell}{2}\sin\theta\, d\theta = \frac{2\ell}{\pi d}
\end{aligned}$$

What is a derived distribution

- It is a pmf or pdf of a function of a random variable with known probability law. E.g.



  - Obtaining the PDF for

    $$g(X,Y) = Y/X$$

    involves deriving a distribution.
    Note: $g(X,Y)$ is a random variable

When not to find them

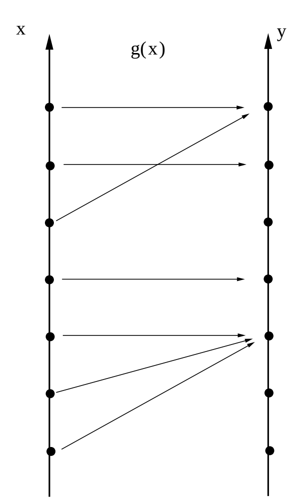
- Don't need PDF for $g(X,Y)$ if only want to compute expected value:

$$\mathbf{E}[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

How to find them

- **Discrete case**
  - Obtain probability mass for each possible value of $Y = g(X)$

$$\begin{aligned}
p_Y(y) &= \mathbf{P}(g(X) = y) \\
&= \sum_{x:\, g(x) = y} p_X(x)
\end{aligned}$$



- **Two-step procedure for the continuous case:**
  - Get CDF of Y: $F_Y(y) = \mathbf{P}(Y \le y)$
  - Differentiate to get
$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

Example

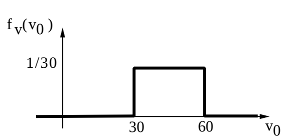
- X: uniform on [0,2]
- Find PDF of $Y = X^3$
- **Solution:**

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) = \mathbf{P}(X^3 \le y) \\ &= \mathbf{P}(X \le y^{1/3}) = \frac{1}{2}y^{1/3} \end{aligned}$$

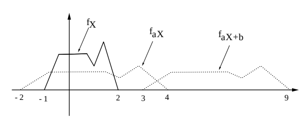
$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{6y^{2/3}}$$

Example

- Joan is driving from Boston to New York. Her speed is uniformly distributed between 30 and 60 mph. What is the distribution of the duration of the trip?
- Let $T(V) = \dfrac{200}{V}$.
- Find $f_T(t)$



The pdf of Y=aX+b



$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Use this to check that if X is normal, then $Y = aX + b$ is also normal.