

LECTURE 13

- **Readings:** Sections 4.3, 4.4

Lecture outline

- Conditional expectation
 - Law of iterated expectations
 - Law of conditional variances
- Sum of a random number of independent r.v.'s
 - mean, variance, transform

Conditional variance

- $\text{Var}(X | Y)$: variance of the conditional distribution of X

$$\text{var}(X | Y = y) = \mathbf{E} \left[(X - \mathbf{E}[X | Y = y])^2 | Y = y \right]$$

- Interesting formula:

$$\text{Var}(X) = \mathbf{E}[\text{Var}(X | Y)] + \text{Var}(\mathbf{E}[X | Y])$$

Conditional expectations

- Given the value y of a r.v. Y :

$$\mathbf{E}[X | Y = y] = \sum_x x p_{X|Y}(x | y)$$

(integral in continuous case)

- Stick example: stick of length ℓ
break at uniformly chosen point Y
break again at uniformly chosen point X

- $\mathbf{E}[X | Y = y] = \frac{y}{2}$ (number)

- $\mathbf{E}[X | Y] = \frac{Y}{2}$ (r.v.)

- **Law of iterated expectations:**

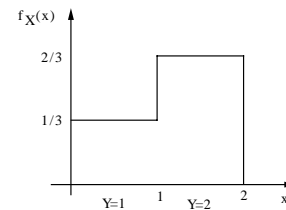
$$\mathbf{E}[\mathbf{E}[X | Y]] = \sum_y \mathbf{E}[X | Y = y] p_Y(y) = \mathbf{E}[X]$$

- In stick example:

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X | Y]] = \mathbf{E}[Y/2] = \ell/4$$

Example

$$\text{Var}(X) = \mathbf{E}[\text{Var}(X | Y)] + \text{Var}(\mathbf{E}[X | Y])$$



$$\mathbf{E}[X | Y = 1] = \quad \mathbf{E}[X | Y = 2] =$$

$$\text{Var}(X | Y = 1) = \quad \text{Var}(X | Y = 2) =$$

$$\mathbf{E}[X] =$$

$$\text{Var}(\mathbf{E}[X | Y]) =$$

Sum of a random number of independent r.v.'s

- N : number of stores visited
- X_i : money spent in store i
 - X_i assumed i.i.d.
 - independent of N

- Let $Y = X_1 + \dots + X_N$

$$\begin{aligned} \mathbf{E}[Y] &= \mathbf{E}[\mathbf{E}[Y | N]] \\ &= \mathbf{E}[N\mathbf{E}[X]] \\ &= \mathbf{E}[N]\mathbf{E}[X] \end{aligned}$$

- Variance:

$$\begin{aligned} \text{Var}(Y) &= \mathbf{E}[\text{Var}(Y | N)] + \text{Var}(\mathbf{E}[Y | N]) \\ &= \mathbf{E}[N]\text{var}(X) + (\mathbf{E}[X])^2\text{var}(N) \end{aligned}$$

Example

$$p_N(n) = \frac{1}{3} \left(\frac{2}{3}\right)^{n-1}, \quad n = 1, 2, \dots$$

$$f_X(x) = 3e^{-3x}, \quad x \geq 0$$

- Find pdf of $Y = X_1 + \dots + X_N$

$$M_N(s) = \frac{e^s/3}{1 - 2e^s/3}, \quad M_X(s) = \frac{3}{3 - s}$$

$$M_Y(s) = \frac{M_X(s)/3}{1 - 2M_X(s)/3} = \frac{1}{1 - s}$$

$$f_Y(y) =$$

Review of transforms

- Definitions:

$$M_X(s) = \mathbf{E}[e^{sX}] = \begin{cases} \sum e^{sx} p_X(x) \\ \int_{-\infty}^{\infty} e^{sx} f_X(x) dx \end{cases}$$

- Moment generating properties:

$$\left. \frac{d^n}{ds^n} M_X(s) \right|_{s=0} = \mathbf{E}[X^n]$$

- Transform of sum of independent r.v.'s X, Y independent; $W = X + Y$

$$M_W(s) = M_X(s)M_Y(s)$$

- Transform of "random sum":

$$\begin{aligned} M_Y(s) &= \mathbf{E}[e^{sY}] \\ &= \mathbf{E}[\mathbf{E}[e^{sY} | N]] \\ &= \mathbf{E}[\mathbf{E}[e^{s(X_1 + \dots + X_N)} | N]] \\ &= \mathbf{E}[M_X(s)^N] \end{aligned}$$

- compare with $M_N(s) = \mathbf{E}[(e^s)^N]$

- start with $M_N(s)$ and replace occurrences of e^s by $M_X(s)$