Recitation 9: Answers
March 10, 2005

1. Following the usual technique for finding the density of a function of a random variable, we first find the distribution, and then differentiate to find the density.
   a) $Y = X^2$.

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) \\ &= \mathbf{P}(X^2 \leq y) \\ &= \mathbf{P}(X \leq \sqrt{y}) \end{aligned}$$

   Thus,

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \sqrt{y} & 0 \leq y \leq 1 \\ 1 & y \geq 1 \end{cases}$$

   and therefore we have:

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   b) $Y = e^X$.

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) \\ &= \mathbf{P}(e^X \leq y) \\ &= \mathbf{P}(X \leq \ln y) \end{aligned}$$

   Thus,

$$F_Y(y) = \begin{cases} 0 & y \leq 1 \\ \ln y & 1 \leq y \leq e \\ 1 & y \geq e \end{cases}$$

   and thus we have:

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 \leq y \leq e \\ 0 & \text{otherwise} \end{cases}$$

2. There are two approaches to this question. First we will present a "pattern matching" approach and then a "derivative" approach.

Pattern Matching Approach. Define the events

$$\begin{aligned} A_{r,\theta} &= \{X^2+Y^2 \le r^2, Y \le X\tan(\theta), X>0\} \text{ and} \\ B_{r,\theta} &= \{X^2+Y^2 \le r^2, Y \ge X\tan(\theta), X<0\}. \end{aligned}$$

The only range of interest is $0 \le r \le 1$ and $-\pi/2 < \theta < \pi/2$, since for any values outside that range, the joint PDF of R and Θ is zero (i.e., R and Θ do not take values outside that range). For this range, we have

$$\begin{aligned} \mathbf{P}(R \le r, \Theta \le \theta) &= \mathbf{P}(\sqrt{X^2+Y^2} \le r, \tan^{-1}(Y/X) \le \theta) \\ &= \mathbf{P}(A_{r,\theta}) + \mathbf{P}(B_{r,\theta}) \\ &= 2 \cdot \mathbf{P}(A_{r,\theta}) \\ &= 2 \cdot \frac{(\text{area of } A_{r,\theta})}{(\text{area of circle of radius 1})} \\ &= \frac{2}{\pi}\int_0^r \int_{-\pi/2}^{\theta} t\, dt d\omega \\ &= \int_0^r \int_{-\pi/2}^{\theta} \frac{2t}{\pi}\, dt d\omega. \end{aligned}$$

The second equality follows from the total probability theorem. [1] The third and fourth equalities follow by (geometric) symmetry and $f_{X,Y}$ being uniform over the circle of radius 1. The second-to-last equality follows from the expression of the area of $A_{r,\theta}$ as an integral in polar coordinates and the last equality follows by moving the constant factors inside the integral. Figure 1 helps to illustrate the geometry behind the arguments above.

Now, by definition, we also have

$$F_{R,\Theta}(r,\theta) = \mathbf{P}(R \le r, \Theta \le \theta) = \int_0^r \int_{-\pi/2}^{\theta} f_{R,\Theta}(t,\omega)\, dt d\omega.$$

Hence, since the CDF uniquely determines the PDF, by matching the two integral expressions we obtain the joint PDF of R and Θ

$$f_{R,\Theta}(r,\theta) = \begin{cases} 2r/\pi, & \text{if } 0 \le r \le 1 \text{ and } -\pi/2 < \theta < \pi/2, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the joint PDF over the region of interest is *independent of* θ*!*

More importantly, note that R *and* Θ *are independent, even though* X *and* Y *are not!*.

To see this, first note that the marginals are

$$f_R(r) = \begin{cases} 2r, & \text{if } 0 \le r \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

[1] Note that the event $\{\tan^{-1}(Y/X) \le \theta, X>0\} = \{Y \le X\tan(\theta), X > 0\}$ and, similarly, $\{\tan^{-1}(Y/X) \le \theta, X < 0\} = \{Y \ge X\tan(\theta), X < 0\}$. This follows because we can apply the tan function to both sides of the inequality $\tan^{-1}(Y/X) \le \theta$ to obtain $Y/X \le \tan(\theta)$, since $\tan(\omega)$ monotonically increases for $-\pi/2 < \omega < \pi/2$. Also for $X < 0$, multiplying both sides of the resulting inequality by X reverses the inequality.

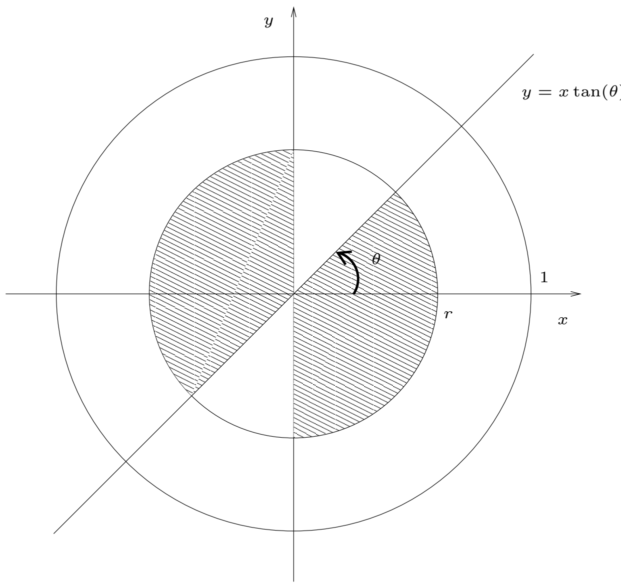

Figure 1: The shaded areas correspond to the event $\{X^2 + Y^2 \le r^2, \tan^{-1}(Y/X) \le \theta\}$.

and

$$f_\Theta(\theta) = \begin{cases} 1/\pi, & \text{if } -\pi/2 < \theta < \pi/2, \\ 0, & \text{otherwise.} \end{cases}$$

(i.e., Θ is uniform), from which we get that

$$f_{R,\Theta}(r,\theta) = f_R(r) f_\Theta(\theta)$$

as claimed.

This problem helps illustrate how our intuitions regarding independence of random variables can break down. One might be tempted to conclude that because both R and Θ are functions of the same, *not independent* random variables X and Y, that R and Θ should not be independent. This need not be the case, as we just found out.

Partial Derivatives Approach. An alternative approach to this problem is to find the joint CDF for R and Θ, and then, to find the joint PDF, take the partial derivatives with respect to r and θ. Note that this means that if for some range of the variables, the CDF is a constant, or a function of only one of the two variables, then it will contribute nothing to the PDF, as it will be wiped out by the appropriate partial derivative. So, as before, we need only consider the case $0 \le r \le 1, -\pi/2 < \theta < \pi/2$, since for any other values of r and θ, the CDF will be constant or a function of only one of the variables. We proceed to find the CDF

$$\begin{aligned} F_{R,\Theta}(r,\theta) &= 2 \cdot \frac{(\text{area of } \{X^2 + Y^2 \le r^2, Y \le X\tan(\theta), X > 0\})}{(\text{area of circle of radius 1})} \\ &= (\theta + \pi/2) r^2/\pi. \end{aligned}$$

The first equality follows from our derivation above and the last equality follows from the geometric argument given above and simplifying the result. [2]

Taking partial derivatives of the joint CDF $F_{R,\Theta}$ with respect to r and θ, we obtain the same joint PDF $f_{R,\Theta}$ as before.

3. (a) $\mathbf{P}(B|A) = \frac{1}{3}$.

   (b) i. $f_{X|Y}(x|0.5) = \begin{cases} 2, & \text{if } 0 < x \leq 1/2, \\ 0, & \text{otherwise.} \end{cases}$

   ii. $E[X|Y = 0.5] = \frac{1}{4}$.

   iii. $\sigma^2_{X|Y=0.5} = \frac{1}{48}$.

   (c) $\mathbf{E}[T] = \frac{1}{12}$.

[2] Note that the area of a "slice" or circular sector having central angle ω and radius r is $\omega r^2/2$.