

Problem Set 7
Due: April 6, 2005

1. Oscar is an electrical engineer, and he is equally likely to work between zero and one hundred hours each week (i.e., the time he works is uniformly distributed between zero and one hundred). He gets paid one dollar an hour.

If Oscar works more than fifty hours during a week, there is a probability of $1/2$ that he will actually be paid overtime, which means he will receive two dollars an hour for each hour he works longer than fifty hours. Otherwise, he will just get his normal pay for all of his hours that week. Independently of receiving overtime pay, if Oscar works more than seventy five hours in a week, there is a probability of $1/2$ that he will receive a one hundred dollar bonus, in addition to whatever else he earns.

Determine the expected value and variance of Oscar's weekly salary.

2. The transform for J , the total number of living groups contacted about the MIT blood drive, is:

$$M_J(s) = \left(\frac{1}{3} + \frac{2}{3}e^s \right)^{10}$$

- (a) Determine the PMF $p_J(j)$.

Let K , the number of people in any particular living group, be an independent random variable with transform:

$$M_K(s) = \frac{\frac{1}{5}e^{4s}}{1 - \frac{4}{5}e^s}$$

- (b) Without doing much work, determine the PMF $p_K(k)$, and evaluate $\mathbf{E}[K]$ and $\text{Var}(K)$.
- (c) Let L be the total number of people whose living groups are contacted about the blood drive. Determine either the PMF or the transform for random variable L . Also evaluate $\mathbf{E}[L]$ and $\text{Var}(L)$.

Suppose the probability that any particular person, whose living group is contacted about the MIT blood drive, donates blood is $\frac{1}{4}$, and that all such individuals make their decisions independently.

- (d) Let M denote the total number of blood donors from the contacted living groups. Determine an explicit numerical expression for $p_M(0)$. Also evaluate $\mathbf{E}[M]$ and $\text{Var}(M)$. (Your expressions, albeit not gracious, should be in closed form.)
3. Suppose four random variables, W , X , Y and Z , are known to have

$$\begin{aligned} \mathbf{E}[W] &= \mathbf{E}[X] = \mathbf{E}[Y] = \mathbf{E}[Z] = 0 \quad , \\ \text{Var}(W) &= \text{Var}(X) = \text{Var}(Y) = \text{Var}(Z) = 1 \quad \text{and,} \end{aligned}$$

furthermore, are known to be pairwise uncorrelated. Given $A = W + X$, $B = X + Y$ and $C = Y + Z$, compute $\mathbf{E}[AB]$ and $\mathbf{E}[AC]$, or the correlation between A & B and A & C , respectively.

4. The value of a random variable X is transmitted, but what we receive (denoted by Y) is the value of X corrupted by some additive noise W ; that is, $Y = X + W$. We know the distribution of X and W , and let us assume that both random variables have the same PDF. Suppose also that X and W are independent.
- (a) Let y be the observed value of Y . Produce the least mean-squares estimate of X given Y , i.e., the estimate \hat{X} of X based on y , such that $\mathbf{E}[(\hat{X} - X)^2 | Y = y]$ is minimized.
 - (b) Suppose X and W had the same distribution, but that they were dependent. What would our new answer be?
5. The receiver is an optical communications system uses a photodetector that counts the number of photons that arrive during the communication session. (The time of the communication session is 1 time unit.)

The sender conveys information by either transmitting or not transmitting photons to the photodetector. The probability of transmitting is p . If she transmits, the number of photons X that she transmits during the session has a Poisson PMF with mean λ per time unit. If she does not transmit, she generates no photons.

Unfortunately, regardless of whether or not she transmits, there may still be photons arriving at the photodetector because of a phenomenon called shot noise. The number N of photons that arrive because of the shot noise has a Poisson PMF with mean μ . N and X are independent. The total number of photons counted by the photodetector is equal to the sum of the transmitted photons and the photons generated by the shot noise effect.

- (a) What is the probability that the sender transmitted if the photodetector counted k photons?
- (b) Before you know anything else about a particular communication session, what is your least squares estimate of the number of photons transmitted?
- (c) What is the least squares estimate of the number of photons transmitted by the sender if the photodetector counted k photons?
- (d) What is the best *linear* predictor for the number of photons transmitted by the sender as a function of k , the number of the detected photons?