Review problems for Chapter 7 - Solutions
Due: no due date

1.

$$\begin{aligned} Y_i &= (0.5)^i X_i \\ T_n &= Y_1 + Y_2 + \cdots + Y_n \\ A_n &= \frac{1}{n} T_n \end{aligned}$$

$$\begin{aligned}
\mathbf{E}[Y_n] &= \mathbf{E}\left[(\frac{1}{2})^n X_n\right] = (\frac{1}{2})^n \mathbf{E}[X_n] = \mathbf{E}[X](\frac{1}{2})^n = 2(\frac{1}{2})^n \\
\text{var}(Y_n) &= \text{var}\left((\frac{1}{2})^n X_n\right) = (\frac{1}{2})^{2n}\text{var}(X_n) = \text{var}(X)(\frac{1}{2})^{2n} = 9(\frac{1}{4})^n \\
\mathbf{E}[T_n] &= \mathbf{E}[Y_1 + Y_2 + \cdots + Y_n] = \mathbf{E}[Y_1] + \mathbf{E}[Y_2] + \cdots + \mathbf{E}[Y_n] \\
&= 2\sum(\frac{1}{2})^i = 2\frac{0.5(1-0.5^n)}{1-0.5} = 2(1 - \left(\frac{1}{2}\right)^n) \\
\text{var}(T_n) &= \text{var}(Y_1 + Y_2 + \cdots + Y_n) = \sum(\frac{1}{4})^i \text{var}(X_i) \\
&= 9\left(\frac{\frac{1}{4}\left(1 - \left(\frac{1}{4}\right)^n\right)}{1 - \frac{1}{4}}\right) = 3\left(1 - \left(\frac{1}{4}\right)^n\right) \\
\mathbf{E}[A_n] &= \mathbf{E}\left[\frac{1}{n}T_n\right] = \frac{1}{n}\mathbf{E}[T_n] = \frac{2}{n}(1 - \left(\frac{1}{2}\right)^n) \\
\text{var}(A_n) &= \text{var}\left(\frac{1}{n}T_n\right) = \left(\frac{1}{n}\right)^2 \text{var}(T_n) = \frac{3}{n^2}\left(1 - \left(\frac{1}{4}\right)^n\right)
\end{aligned}$$

(a) Yes. Y_n converges to 0 in probability. As n becomes very large, the expected value of Y_n approaches 0 and the variance of Y_n approaches 0. So, by the Chebychev Inequality, Y_n converges to 0 in probability.

(b) No. Assume that T_n converges in probability to some value a. We also know that:

$$\begin{aligned}
T_n &= Y_1 + (Y_2 + Y_3 + ...Y_n) \\
&= Y_1 + (.5^2 * X_2 + .5^3 * X_3 + ... + .5^n * X_n) \\
&= Y_1 + \frac{1}{2}(.5 * X_2 + .5^2 * X_3 + ... + .5^{n-1} * X_n).
\end{aligned}$$

Notice that $.5 * X_2 + .5^2 * X_3 + ... + .5^{n-1} * X_n$ converges to the same limit as T_n when n goes to infinity. If T_n is to converge to a, Y_1 must converge to $a/2$. But this is clearly false, which presents a contradiction in our original assumption.

(c) Yes. A_n converges to 0 in probability. As n becomes very large, the expected value of A_n approaches 0, and the variance of A_n approaches 0. So, by the Chebychev Inequality, A_n converges to 0 in probability. You could also show this by noting that the A_n's are i.i.d. with finite mean and variance, and using the WLLN.

2. (a) Let X_i be random variables indicating the quality of the ith bulb ("1" for good bulbs, "0" for bad ones). Then X_i are independent Bernoulli random variables. Let Z_n be

$$Z_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

We apply the Chebyshev inquality and obtain

$$\mathbf{P}\left(|Z_n - p| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2},$$

where σ^2 is the variance of the Bernoulli random variable. Hence, we obtain

$$\lim_{n\to\infty} \mathbf{P}\left(|Z_n - p| \geq \epsilon\right) = 0,$$

by noticing $\lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} = 0$. This means that Z_n converges to p in probability.

(b) For any number greater than 500, we know the number of bulbs would be enough for the test by using Chebyshev. Since the variance of a Bernoulli random variable is $p(1-p)$ which is less than or equal to $\frac{1}{4}$, we have $\sigma^2 \leq \frac{1}{4}$. Hence, for $n \geq 500$,

$$\begin{aligned}
\mathbf{P}\left(\left|\frac{X_1 + X_2 + ... + X_n}{n} - p\right| \geq 0.1\right) &\leq \frac{\sigma^2}{n0.1^2} \\
&\leq \frac{\frac{1}{4}}{n \times 0.1^2} \\
&\leq 1 - 0.95 = 0.05.
\end{aligned}$$

However, for a number less than 500, we can not tell if the number of bulbs is enough for the test because we don't know the variance. If the variance is very small, which is possible when p is quite small, 27 bulbs could be enough actually.
Thus, the answer is "cannot be decided". In reality, we need to estimate the variance first.

3. Since the probability of winning is close to .5, the normal approximation to the binomial is quite appropriate. Letting N be the number of wins, we have:

$$\mathbf{E}[N] = 100 \cdot \frac{18}{37} = 48.6, \qquad \text{var}(N) = 100 \cdot \frac{18}{37}\Big(1 - \frac{18}{37}\Big) = 24.98, \qquad \sigma_N = \sqrt{24.98} = 4.998.$$

Hence,

$$\begin{aligned}
\mathbf{P}(N \geq 50) &= \mathbf{P}(N \geq 49.5) \\
&= \mathbf{P}\Big((N - 48.6)/4.998 \geq (49.5 - 48.6)/4.998\Big) \\
&= \mathbf{P}\Big((N - 48.6)/4.998 \geq 0.18\Big) \\
&= 1 - \Phi(0.18) \\
&= 1 - 0.5714 \\
&= 0.4286.
\end{aligned}$$

4. See the solution on Page 403.